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Abstract [Invariant differential operators on Siegel-Jacobi space and Maass-Jacobi forms]
For two positive integers m and n, we let $\mathbb{H}_n$ be the Siegel upper half plane of degree n and let $\mathbb{C}^{(m,n)}$ be the set of all $m \times n$ complex matrices. In this article, we study differential operators on the Siegel-Jacobi space $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$ that are invariant under the *natural* action of the Jacobi group $Sp(n, \mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)}$ on $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$, where $H_{\mathbb{R}}^{(n,m)}$ denotes the Heisenberg group. We give some explicit invariant differential operators. We present important problems which are natural. We announce some solutions for these natural problems. Finally we introduce a notion of Maass-Jacobi forms.

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Invariant differential operators on Siegel-Jacobi space and Maass-Jacobi forms

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1 Introduction

For a given fixed positive integer n, we let

$$\mathbb{H}_n = \{\, \Omega \in \mathbb{C}^{(n,n)} \mid \Omega = {}^t\Omega, \quad \mathrm{Im}\,\Omega > 0\,\}$$

be the Siegel upper half plane of degree n and let

$$Sp(n,\mathbb{R}) = \{ M \in \mathbb{R}^{(2n,2n)} \mid {}^tM J_n M = J_n \}$$

be the symplectic group of degree n, where $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F for two positive integers k and l, ${}^tM$ denotes the transpose of a matrix M and

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

$Sp(n,\mathbb{R})$ acts on $\mathbb{H}_n$ transitively by

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$$M \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1}, \tag{1.1}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$ and $\Omega \in \mathbb{H}_n$.

For two positive integers m and n, we consider the Heisenberg group

$$H_{\mathbb{R}}^{(n,m)} = \left\{ (\lambda, \mu; \kappa) \mid \lambda, \mu \in \mathbb{R}^{(m,n)},\ \kappa \in \mathbb{R}^{(m,m)},\ \kappa + \mu\, {}^t\lambda \text{ symmetric } \right\}$$

endowed with the following multiplication law

$$\big(\lambda, \mu; \kappa\big) \circ \big(\lambda', \mu'; \kappa'\big) = \big(\lambda + \lambda', \mu + \mu'; \kappa + \kappa' + \lambda\, {}^t\mu' - \mu\, {}^t\lambda'\big)$$

with $\big(\lambda, \mu; \kappa\big), \big(\lambda', \mu'; \kappa'\big) \in H_{\mathbb{R}}^{(n,m)}$. We define the *Jacobi group* of degree n and index m that is the semidirect product of $Sp(n, \mathbb{R})$ and $H_{\mathbb{R}}^{(n,m)}$

$$G^J = Sp(n, \mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)}$$

endowed with the following multiplication law

$$\big(M, (\lambda, \mu; \kappa)\big) \cdot \big(M', (\lambda', \mu'; \kappa'\,)\big) = \big(MM', (\tilde{\lambda} + \lambda', \tilde{\mu} + \mu'; \kappa + \kappa' + \tilde{\lambda}\, {}^t\mu' - \tilde{\mu}\, {}^t\lambda'\,)\big)$$

with $M, M' \in Sp(n, \mathbb{R}), (\lambda, \mu; \kappa),\ (\lambda', \mu'; \kappa') \in H_{\mathbb{R}}^{(n,m)}$ and $(\tilde{\lambda}, \tilde{\mu}) = (\lambda, \mu)M'$. Then G^J acts on $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$ transitively by

$$\big(M, (\lambda, \mu; \kappa)\big) \cdot (\Omega, Z) = \Big(M \cdot \Omega, (Z + \lambda\Omega + \mu)(C\Omega + D)^{-1}\Big), \tag{1.2}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R}),\ (\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(n,m)}$ and $(\Omega, Z) \in \mathbb{H}_n \times \mathbb{C}^{(m,n)}$. We note that the Jacobi group G^J is *not* a reductive Lie group and the homogeneous space $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$ is not a symmetric space. We refer to [2, 7, 22, 23, 24, 25, 27, 28, 29, 30, 31] about automorphic forms on G^J and topics related to the content of this paper. From now on, for brevity we write $\mathbb{H}_{n,m} = \mathbb{H}_n \times \mathbb{C}^{(m,n)}$. The homogeneous space $\mathbb{H}_{n,m}$ is called the *Siegel-Jacobi space* of degree n and index m.

The aim of this survey paper is to present results on differential operators on $\mathbb{H}_{n,m}$ which are invariant under the *natural* action (1.2) of G^J. The study of these invariant differential operators on the Siegel-Jacobi space $\mathbb{H}_{n,m}$ is interesting and important in the aspects of invariant theory, arithmetic and geometry. This article is organized as follows. In Section 2, we review differential operators on $\mathbb{H}_n$ invariant under the action (1.1) of $Sp(n, \mathbb{R})$. In Section 3, we discuss differential operators on $\mathbb{H}_{n,m}$ invariant under the action (1.2) of G^J and propose some natural problems related to invariant differential operators on the Siegel-Jacobi space. We present some results without proofs. In Section 4, we gives some examples of explicit G^J-invariant differential operators on $\mathbb{H}_{n,m}$. In Section 5, we introduce the partial Cayley transform of the Siegel-Jacobi space into the Siegel-Jacobi disk and present some explicit invariant differential operators on the Siegel-Jacobi disk. In Section 6, we present some results in the special case $n = m = 1$ in detail. We give complete solutions of the problems that are proposed in Section 3. In the final section, using these invariant differential operators on the Siegel-Jacobi space, we discuss a notion of Maass-Jacobi forms.

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Notations: We denote by $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the field of rational numbers, the field of real numbers and the field of complex numbers respectively. We denote by $\mathbb{Z}$ and $\mathbb{Z}^+$ the ring of integers and the set of all positive integers respectively. The symbol ":=" means that the expression on the right is the definition of that on the left. For two positive integers k and l, $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F. For a square matrix $A \in F^{(k,k)}$ of degree k, $\mathrm{tr}(A)$ denotes the trace of A. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transpose of a matrix M. For $A \in F^{(k,l)}$ and $B \in F^{(k,k)}$, we set $B[A] = {}^tABA$. For a positive integer n, I_n denotes the identity matrix of degree n. For a complex number z, $|z|$ denotes the absolute value of z. For a complex number z, $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$ denote the real part of z and the imaginary part of z respectively.

2 Invariant Differential Operators on Siegel Space

For a coordinate $\Omega = (\omega_{ij}) \in \mathbb{H}_n$, we write $\Omega = X + i\,Y$ with $X = (x_{ij}),\ Y = (y_{ij})$ real. We put $d\Omega = \big(d\omega_{ij}\big)$ and $d\overline{\Omega} = \big(d\overline{\omega}_{ij}\big)$. We also put

$$\frac{\partial}{\partial \Omega} = \left(\frac{1+\delta_{ij}}{2}\, \frac{\partial}{\partial \omega_{ij}} \right) \qquad \text{and} \qquad \frac{\partial}{\partial \overline{\Omega}} = \left(\frac{1+\delta_{ij}}{2}\, \frac{\partial}{\partial \overline{\omega}_{ij}} \right).$$

Then for a positive real number A,

$$ds^2_{n;A} = A\, \mathrm{tr}\Big(Y^{-1} d\Omega\, Y^{-1} d\overline{\Omega} \Big) \tag{2.1}$$

is a $Sp(n,\mathbb{R})$-invariant Kähler metric on $\mathbb{H}_n$ (cf. [19, 20]), where $\mathrm{tr}(M)$ denotes the trace of a square matrix M. H. Maass [14] proved that the Laplacian of $ds^2_{n;A}$ is given by

$$\Delta_{n;A} = \frac{4}{A}\,\mathrm{tr}\left(Y\,{}^t\!\left(Y\frac{\partial}{\partial\overline{\Omega}}\right)\frac{\partial}{\partial\Omega}\right). \tag{2.2}$$

And

$$dv_n(\Omega) = (\det Y)^{-(n+1)} \prod_{1\le i\le j\le n} dx_{ij} \prod_{1\le i\le j\le n} dy_{ij}$$

is a $Sp(n,\mathbb{R})$-invariant volume element on $\mathbb{H}_n$ (cf. [20, p. 130]).

For brevity, we write $G = Sp(n,\mathbb{R})$. The isotropy subgroup K at iI_n for the action (1.1) is a maximal compact subgroup of G given by

$$K = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \;\Big|\; A\,{}^tA + B\,{}^tB = I_n,\ A\,{}^tB = B\,{}^tA,\ A, B \in \mathbb{R}^{(n,n)} \right\}.$$

Let $\mathfrak{k}$ be the Lie algebra of K. Then the Lie algebra $\mathfrak{g}$ of G has a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where

$$\mathfrak{g} = \left\{ \begin{pmatrix} X_1 & X_2 \\ X_3 & -\,{}^tX_1 \end{pmatrix} \;\Big|\; X_1, X_2, X_3 \in \mathbb{R}^{(n,n)},\ X_2 = {}^tX_2,\ X_3 = {}^tX_3 \right\},$$

$$\mathfrak{k} = \left\{ \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in \mathbb{R}^{(2n,2n)} \;\Big|\; {}^tX + X = 0,\ Y = {}^tY \right\},$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \;\Big|\; X = {}^tX,\ Y = {}^tY,\ X, Y \in \mathbb{R}^{(n,n)} \right\}.$$

The subspace $\mathfrak{p}$ of $\mathfrak{g}$ may be regarded as the tangent space of $\mathbb{H}_n$ at iI_n. The adjoint representation of G on $\mathfrak{g}$ induces the action of K on $\mathfrak{p}$ given by

$$k \cdot Z = kZ\,{}^tk, \quad k \in K,\ Z \in \mathfrak{p}. \tag{2.3}$$

Let T_n be the vector space of $n \times n$ symmetric complex matrices. We let $\Psi : \mathfrak{p} \longrightarrow T_n$ be the map defined by

$$\Psi\left(\begin{pmatrix} X & Y \\ Y & -X \end{pmatrix}\right) = X + i\,Y, \quad \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \in \mathfrak{p}. \tag{2.4}$$

We let $\delta : K \longrightarrow U(n)$ be the isomorphism defined by

$$\delta\left(\begin{pmatrix} A & -B \\ B & A \end{pmatrix}\right) = A + i\,B, \quad \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in K, \tag{2.5}$$

where $U(n)$ denotes the unitary group of degree n. We identify $\mathfrak{p}$ (resp. K) with T_n (resp. $U(n)$) through the map Ψ (resp. δ). We consider the action of $U(n)$ on T_n defined by

$$h \cdot \omega = h\omega\,{}^th, \quad h \in U(n),\ \omega \in T_n. \tag{2.6}$$

Then the adjoint action (2.3) of K on $\mathfrak{p}$ is compatible with the action (2.6) of $U(n)$ on T_n through the map Ψ. Precisely for any $k \in K$ and $Z \in \mathfrak{p}$, we get

$$\Psi(k\,Z\,{}^t k) = \delta(k)\,\Psi(Z)\,{}^t\delta(k). \tag{2.7}$$

The action (2.6) induces the action of $U(n)$ on the polynomial algebra $\mathrm{Pol}(T_n)$ consisting of polynomials on T_n. We denote by $\mathrm{Pol}(T_n)^{U(n)}$ the subalgebra of $\mathrm{Pol}(T_n)$ consisting of polynomials invariant under the action of $U(n)$. Then we have the so-called Helgason map

$$\Theta_n : \mathrm{Pol}(T_n)^{U(n)} \longrightarrow \mathbb{D}(\mathbb{H}_n) \tag{2.8}$$

of $\mathrm{Pol}(T_n)^{U(n)}$ onto the algebra $\mathbb{D}(\mathbb{H}_n)$ of differential operators on $\mathbb{H}_n$ invariant under the action (1.1) of G. The map Θ_n is a canonical linear bijection but is not an algebra isomorphism. The map Θ_n is described explicitly as follows. We put $N = n(n+1)$. Let $\{\xi_\alpha \mid 1 \le \alpha \le N\}$ be a basis of a real vector space $\mathfrak{p}$. If $P \in \mathrm{Pol}(\mathfrak{p})^K$, then

$$\Big(\Theta_n(P)f\Big)(gK) = \left[P\left(\frac{\partial}{\partial t_\alpha}\right) f\left(g\exp\left(\sum_{\alpha=1}^{N} t_\alpha \xi_\alpha\right) K\right)\right]_{(t_\alpha)=0}, \tag{2.9}$$

where $f \in C^\infty(\mathbb{H}_n)$. We refer to [10, 11] for more detail. In general, it is hard to express $\Phi(P)$ explicitly for a polynomial $P \in \mathrm{Pol}(\mathfrak{p})^K$.

According to the work of Harish-Chandra [8, 9], the algebra $\mathbb{D}(\mathbb{H}_n)$ is generated by n algebraically independent generators and is isomorphic to the commutative algebra $\mathbb{C}[x_1, \cdots, x_n]$ with n indeterminates. We note that n is the real rank of G. Let $\mathfrak{g}_{\mathbb{C}}$ be the complexification of $\mathfrak{g}$. It is known that $\mathbb{D}(\mathbb{H}_n)$ is isomorphic to the center of the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$.

Using a classical invariant theory (cf. [12, 21], we can show that $\mathrm{Pol}(T_n)^{U(n)}$ is generated by the following algebraically independent polynomials

$$q_j(\omega) = \mathrm{tr}\Big(\big(\omega\overline{\omega}\big)^j\Big), \quad \omega \in T_n, \quad j = 1, 2, \cdots, n. \tag{2.10}$$

For each j with $1 \le j \le n$, the image $\Theta_n(q_j)$ of q_j is an invariant differential operator on $\mathbb{H}_n$ of degree $2j$. The algebra $\mathbb{D}(\mathbb{H}_n)$ is generated by n algebraically independent generators $\Theta_n(q_1), \Theta_n(q_2), \cdots, \Theta_n(q_n)$. In particular,

$$\Theta_n(q_1) = c_1\,\mathrm{tr}\left(Y\,{}^t\!\left(Y \frac{\partial}{\partial \overline{\Omega}}\right) \frac{\partial}{\partial \Omega}\right) \quad \text{for some constant } c_1. \tag{2.11}$$

We observe that if we take $\omega = x + i\,y \in T_n$ with real x, y, then $q_1(\omega) = q_1(x, y) = \mathrm{tr}\big(x^2 + y^2\big)$ and

$$q_2(\omega) = q_2(x,y) = \,\mathrm{tr}\Big(\big(x^2+y^2\big)^2 + \,2\,x\big(xy-yx\big)y\,\Big).$$

It is a natural question to express the images $\Theta_n(q_j)$ explicitly for $j = 2,3,\cdots,n$. We hope that the images $\Theta_n(q_j)$ for $j = 2,3,\cdots,n$ are expressed in the form of the *trace.*

H. Maass [15] found explicit algebraically independent generators of $\mathbb{D}(\mathbb{H}_n)$. G. Shimura [18] found canonically defined algebraically independent generators of $\mathbb{D}(\mathbb{H}_n)$.

Example 2.1. We consider the case $n = 1$. The algebra $\mathrm{Pol}(T_1)^{U(1)}$ is generated by the polynomial

$$q(\omega) = \omega\,\overline{\omega}, \quad \omega = x + \,iy \in \mathbb{C} \text{ with } x,y \text{ real}.$$

Using Formula (2.9), we get

$$\Theta_1(q) = \,4\,y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

Therefore $\mathbb{D}(\mathbb{H}_1) = \mathbb{C}\big[\Theta_1(q)\big]$.

Example 2.2. We consider the case $n = 2$. The algebra $\mathrm{Pol}(T_2)^{U(2)}$ is generated by the polynomial

$$q_1(\omega) = \,\mathrm{tr}\big(\omega\,\overline{\omega}\,\big), \quad q_2(\omega) = \,\mathrm{tr}\Big(\big(\omega\,\overline{\omega}\big)^2\Big), \quad \omega \in T_2.$$

Using Formula (2.9), we may express $\Theta_2(q_1)$ and $\Theta_2(q_2)$ explicitly. $\Theta_2(q_1)$ is expressed by Formula (2.11). The computation of $\Theta_2(q_2)$ might be quite tedious. We leave the detail to the reader. In this case, $\Theta_2(q_2)$ was essentially computed in [5], Proposition 6. Therefore

$$\mathbb{D}(\mathbb{H}_2) = \mathbb{C}\big[\Theta_2(q_1), \Theta_2(q_2)\big].$$

In fact, the center of the universal enveloping algebra $\mathscr{U}(\mathfrak{g}_{\mathbb{C}})$ was computed in [5].

3 Invariant Differential Operators on Siegel-Jacobi Space

The stabilizer K^J of G^J at $(iI_n, 0)$ is given by

$$K^J = \Big\{ \big(k, (0,0;\kappa)\big) \,\big|\, k \in K,\ \kappa = {}^t\kappa \in \mathbb{R}^{(m,m)} \Big\}.$$

Therefore $\mathbb{H}_{n,m} \cong G^J/K^J$ is a homogeneous space of *non-reductive type*. The Lie algebra $\mathfrak{g}^J$ of G^J has a decomposition

$$\mathfrak{g}^J = \mathfrak{k}^J + \mathfrak{p}^J,$$

where

$$\mathfrak{g}^J = \Big\{ \big(Z, (P,Q,R)\big) \,\big|\, Z \in \mathfrak{g},\ P,Q \in \mathbb{R}^{(m,n)},\ R = {}^tR \in \mathbb{R}^{(m,m)} \Big\},$$

$$\mathfrak{k}^J = \Big\{ \big(X, (0,0,R)\big) \,\big|\, X \in \mathfrak{k},\ R = {}^tR \in \mathbb{R}^{(m,m)} \Big\},$$

$$\mathfrak{p}^J = \Big\{ \big(Y, (P,Q,0)\big) \,\big|\, Y \in \mathfrak{p},\ P,Q \in \mathbb{R}^{(m,n)} \Big\}.$$

Thus the tangent space of the homogeneous space $\mathbb{H}_{n,m}$ at $(iI_n, 0)$ is identified with $\mathfrak{p}^J$.

If $\alpha = \left(\begin{pmatrix} X_1 & Y_1 \\ Z_1 & -{}^tX_1 \end{pmatrix}, (P_1, Q_1, R_1) \right)$ and $\beta = \left(\begin{pmatrix} X_2 & Y_2 \\ Z_2 & -{}^tX_2 \end{pmatrix}, (P_2, Q_2, R_2) \right)$ are elements of $\mathfrak{g}^J$, then the Lie bracket $[\alpha, \beta]$ of α and β is given by

$$[\alpha,\beta] = \left(\begin{pmatrix} X^* & Y^* \\ Z^* & -{}^tX^* \end{pmatrix}, (P^*, Q^*, R^*) \right), \tag{3.1}$$

where

$$\begin{aligned}
X^* &= X_1X_2 - X_2X_1 + Y_1Z_2 - Y_2Z_1, \\
Y^* &= X_1Y_2 - X_2Y_1 + Y_2\,{}^tX_1 - Y_1\,{}^tX_2, \\
Z^* &= Z_1X_2 - Z_2X_1 + {}^tX_2Z_1 - {}^tX_1Z_2, \\
P^* &= P_1X_2 - P_2X_1 + Q_1Z_2 - Q_2Z_1, \\
Q^* &= P_1Y_2 - P_2Y_1 + Q_2\,{}^tX_1 - Q_1\,{}^tX_2, \\
R^* &= P_1\,{}^tQ_2 - P_2\,{}^tQ_1 + Q_2\,{}^tP_1 - Q_1\,{}^tP_2.
\end{aligned}$$

We recall that T_n denotes the vector space of all $n \times n$ symmetric complex matrices. For brevity, we put $T_{n,m} := T_n \times \mathbb{C}^{(m,n)}$. We define the real linear map $\Phi : \mathfrak{p}^J \longrightarrow T_{n,m}$ by

$$\Phi\left(\begin{pmatrix} X & Y \\ Y & -X \end{pmatrix}, (P,Q,0) \right) = \big(X + i\,Y,\ P + i\,Q\big), \tag{3.2}$$

where $\begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \in \mathfrak{p}$ and $P,Q \in \mathbb{R}^{(m,n)}$.

Let $S(m,\mathbb{R})$ denote the additive group consisting of all $m\times m$ real symmetric matrices. Now we define the isomorphism $\theta : K^J \longrightarrow U(n)\times S(m,\mathbb{R})$ by

$$\theta(h,(0,0,\kappa)) = (\delta(h),\kappa),\quad h\in K,\ \kappa\in S(m,\mathbb{R}), \tag{3.3}$$

where $\delta : K\longrightarrow U(n)$ is the map defined by (2.5). Identifying $\mathbb{R}^{(m,n)}\times\mathbb{R}^{(m,n)}$ with $\mathbb{C}^{(m,n)}$, we can identify $\mathfrak{p}^J$ with $T_n\times\mathbb{C}^{(m,n)}$.

Theorem 3.1. *The adjoint representation of K^J on $\mathfrak{p}^J$ is compatible with the natural action of $U(n)\times S(m,\mathbb{R})$ on $T_{n,m}$ defined by*

$$(h,\kappa)\cdot(\omega,z) := (h\,\omega\,{}^t h,\ z\,{}^t h) \tag{3.4}$$

through the maps Φ and θ, where $h\in U(n)$, $\kappa\in S(m,\mathbb{R})$, $(\omega,z)\in T_{n,m}$. Precisely, if $k^J\in K^J$ and $\alpha\in\mathfrak{p}^J$, then we have the following equality

$$\Phi\big(Ad\big(k^J\big)\alpha\big) = \theta\big(k^J\big)\cdot\Phi(\alpha). \tag{3.5}$$

Here we regard the complex vector space $T_{n,m}$ as a real vector space.

The proof of the above theorem can be found in [13].

We now study the algebra $\mathbb{D}(\mathbb{H}_{n,m})$ of all differential operators on $\mathbb{H}_{n,m}$ invariant under the *natural action* (1.2) of G^J. The action (3.4) induces the action of $U(n)$ on the polynomial algebra $\mathrm{Pol}_{n,m} := \mathrm{Pol}\,(T_{n,m})$. We denote by $\mathrm{Pol}_{n,m}^{U(n)}$ the subalgebra of $\mathrm{Pol}_{n,m}$ consisting of all $U(n)$-invariants. Similarly the adjoint action of K on $\mathfrak{p}^J$ induces the action of K on the polynomial algebra $\mathrm{Pol}\big(\mathfrak{p}^J\big)$. We see that through the identification of $\mathfrak{p}^J$ with $T_{n,m}$, the algebra $\mathrm{Pol}\big(\mathfrak{p}^J\big)$ is isomorphic to $\mathrm{Pol}_{n,m}$.

According to Helgason ([11], p. 287), one obtains the Helgason map

$$\Theta_{n,m} : \mathrm{Pol}_{n,m}^{U(n)} \longrightarrow \mathbb{D}(\mathbb{H}_{n,m})$$

of $\mathrm{Pol}_{n,m}^{U(n)}$ onto $\mathbb{D}(\mathbb{H}_{n,m})$ which is a natural linear bijection but is not an algebra isomorphism. The map $\Theta_{n,m}$ is described explicitly as follows. We put $N_\star = n(n+1)+2mn$. Let $\big\{\eta_\alpha \mid 1\leq\alpha\leq N_\star\big\}$ be a basis of $\mathfrak{p}^J$. If $P\in\mathrm{Pol}\big(\mathfrak{p}^J\big)^K = \mathrm{Pol}_{n,m}^{U(n)}$, then

$$\Big(\Theta_{n,m}(P)f\Big)(gK^J) = \left[P\left(\frac{\partial}{\partial t_\alpha}\right) f\left(g\exp\left(\sum_{\alpha=1}^{N_\star} t_\alpha\eta_\alpha\right)K^J\right)\right]_{(t_\alpha)=0}, \tag{3.6}$$

where $g\in G^J$ and $f\in C^\infty(\mathbb{H}_{n,m})$. In general, it is hard to express $\Theta_{n,m}(P)$ explicitly for a polynomial $P\in\mathrm{Pol}\big(\mathfrak{p}^J\big)^K$.

We propose the following natural problems.

Problem 1. Find a complete list of explicit generators of $\mathrm{Pol}_{n,m}^{U(n)}$.

Problem 2. Find all relations among a set of generators of $\mathrm{Pol}_{n,m}^{U(n)}$.

Problem 3. Find an easy or effective way to express explicitly the images of the above invariant polynomials or generators of $\mathrm{Pol}_{n,m}^{U(n)}$ under the Helgason map $\Theta_{n,m}$.

Problem 4. Decompose $\mathrm{Pol}_{n,m}$ into $U(n)$-irreducibles.

Problem 5. Find a complete list of explicit generators of the algebra $\mathbb{D}(\mathbb{H}_{n,m})$ or construct explicit G^J-invariant differential operators on $\mathbb{H}_{n,m}$.

Problem 6. Find all relations among a set of generators of $\mathbb{D}(\mathbb{H}_{n,m})$.

Problem 7. Is $\mathrm{Pol}_{n,m}^{U(n)}$ finitely generated ?

Problem 8. Is $\mathbb{D}(\mathbb{H}_{n,m})$ finitely generated ?

Problem 1 and Problem 7 are solved as follows.

Theorem 3.2. $\mathrm{Pol}_{n,m}^{U(n)}$ *is generated by*

$$\begin{aligned}
q_j(\omega,z) &= \mathrm{tr}\big((\omega\,\overline{\omega})^{j+1}\big), \quad 0\le j\le n-1,\\
\alpha_{kp}^{(j)}(\omega,z) &= \mathrm{Re}\,\big(z\,(\overline{\omega}\omega)^j\,{}^t\overline{z}\big)_{kp}, \quad 0\le j\le n-1,\ 1\le k\le p\le m,\\
\beta_{lq}^{(j)}(\omega,z) &= \mathrm{Im}\,\big(z\,(\overline{\omega}\omega)^j\,{}^t\overline{z}\big)_{lq}, \quad 0\le j\le n-1,\ 1\le l< q\le m,\\
f_{kp}^{(j)}(\omega,z) &= \mathrm{Re}\,(z\,(\overline{\omega}\omega)^j\,\overline{\omega}\,{}^tz)_{kp}, \quad 0\le j\le n-1,\ 1\le k\le p\le m,\\
g_{kp}^{(j)}(\omega,z) &= \mathrm{Im}\,(z\,(\overline{\omega}\omega)^j\,\overline{\omega}\,{}^tz\,)_{kp}, \quad 0\le j\le n-1,\ 1\le k\le p\le m,
\end{aligned}$$

where $\omega\in T_n$ *and* $z\in\mathbb{C}^{(m,n)}$.

The proof of Theorem 3.2 can be found in [13]. Here we will not describe the solution of Problem 2 because it is very complicated. The solution of Problem 2 will appear in another paper in the near future.

4 Examples of Explicit G^J-Invariant Differential Operators

In this section we give examples of explicit G^J-invariant differential operators on the Siegel-Jacobi space $\mathbb{H}_{n,m}$.

For $g = \big(M,(\lambda,\mu;\kappa)\big) \in G^J$ with $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R})$ and $(\Omega, Z) \in \mathbb{H}_{n,m}$, we set

$$\begin{aligned} \Omega &= X + i\,Y, \quad X = (x_{\mu\nu}),\ Y = (y_{\mu\nu}) \text{ real},\\ \Omega_* &= M\cdot\Omega = X_* + i\,Y_*, \quad X_*, Y_* \text{ real},\\ Z_* &= (Z + \lambda\Omega + \mu)(C\Omega + D)^{-1} = U_* + i\,V_*, \quad U_*, V_* \text{ real}. \end{aligned}$$

For a coordinate $(\Omega, Z) \in \mathbb{H}_{n,m}$ with $\Omega = (\omega_{\mu\nu})$ and $Z = (z_{kl})$, we put $d\Omega$, $d\overline{\Omega}$, $\frac{\partial}{\partial\Omega}$, $\frac{\partial}{\partial\overline{\Omega}}$ as before and set

$$\begin{aligned} Z &= U + iV, \quad U = (u_{kl}), \quad V = (v_{kl}) \ \text{ real},\\ dZ &= (dz_{kl}), \quad d\overline{Z} = (d\overline{z}_{kl}), \end{aligned}$$

$$\frac{\partial}{\partial Z} = \begin{pmatrix} \frac{\partial}{\partial z_{11}} & \dots & \frac{\partial}{\partial z_{m1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial z_{1n}} & \dots & \frac{\partial}{\partial z_{mn}} \end{pmatrix}, \quad \frac{\partial}{\partial \overline{Z}} = \begin{pmatrix} \frac{\partial}{\partial \overline{z}_{11}} & \dots & \frac{\partial}{\partial \overline{z}_{m1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial \overline{z}_{1n}} & \dots & \frac{\partial}{\partial \overline{z}_{mn}} \end{pmatrix}.$$

The author [29] proved that the following differential operators $\mathbb{M}_1$ and $\mathbb{M}_2$ on $\mathbb{H}_{n,m}$ defined by

$$\mathbb{M}_1 = \mathrm{tr}\left(Y\,\frac{\partial}{\partial Z}\,{}^t\!\left(\frac{\partial}{\partial \overline{Z}}\right)\right) \tag{4.1}$$

and

$$\begin{aligned} \mathbb{M}_2 = &\ \mathrm{tr}\left(Y\,{}^t\!\left(Y\frac{\partial}{\partial\overline{\Omega}}\right)\frac{\partial}{\partial\Omega}\right) + \mathrm{tr}\left(VY^{-1}\,{}^tV\,{}^t\!\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial Z}\right) \\ &+ \mathrm{tr}\left(V\,{}^t\!\left(Y\frac{\partial}{\partial\overline{\Omega}}\right)\frac{\partial}{\partial Z}\right) + \mathrm{tr}\left({}^tV\,{}^t\!\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial\Omega}\right) \end{aligned} \tag{4.2}$$

are invariant under the action (1.2) of G^J.

The authors [13] proved that the following differential operator $\mathbb{K}$ on $\mathbb{H}_{n,m}$ of degree $2n$ defined by

$$\mathbb{M}_3 = \det(Y)\,\det\left(\frac{\partial}{\partial Z}\,{}^t\!\left(\frac{\partial}{\partial\overline{Z}}\right)\right) \tag{4.3}$$

is invariant under the action (1.2) of G^J. Furthermore the authors [13] proved that the following matrix-valued differential operator $\mathbb{T}$ on $\mathbb{H}_{n,m}$ defined by

$$\mathbb{T} = {}^t\!\left(\frac{\partial}{\partial\overline{Z}}\right) Y \frac{\partial}{\partial Z} \tag{4.4}$$

and the following differential operators

$$\mathbb{T}_{kl} = \sum_{i,j=1}^{n} y_{ij}\, \frac{\partial^2}{\partial \overline{z}_{ki} \partial z_{lj}}, \quad 1 \leq k, l \leq m \tag{4.5}$$

are invariant under the action (1.2) of G^J.

We see that

$$\mathbb{M}_* = [\mathbb{M}_1, \mathbb{M}_2] = \mathbb{M}_1\mathbb{M}_2 - \mathbb{M}_2\mathbb{M}_1$$

is an invariant differential operator of degree three on $\mathbb{H}_{n,m}$ and

$$\mathbb{P}_{kl} = [\mathbb{M}_3, \mathbb{T}_{kl}] = \mathbb{M}_3\mathbb{T}_{kl} - \mathbb{T}_{kl}\mathbb{M}_3, \quad 1 \leq k, l \leq m$$

is an invariant differential operator of degree $2n+1$ on $\mathbb{H}_{n,m}$.

The author [29] proved that for any two positive real numbers A and B,

$$\begin{aligned} ds^2_{n,m;A,B} = \; & A\, \mathrm{tr}\Big(Y^{-1} d\Omega\, Y^{-1} d\overline{\Omega}\Big) \\ & + B\, \Big\{ \mathrm{tr}\Big(Y^{-1}\, {}^t V\, V\, Y^{-1} d\Omega\, Y^{-1} d\overline{\Omega}\Big) + \mathrm{tr}\Big(Y^{-1}\, {}^t(dZ)\, d\overline{Z}\Big) \\ & - \mathrm{tr}\Big(V\, Y^{-1} d\Omega\, Y^{-1}\, {}^t(d\overline{Z})\Big) - \mathrm{tr}\Big(V\, Y^{-1} d\overline{\Omega}\, Y^{-1}\, {}^t(dZ)\,\Big) \Big\} \end{aligned}$$

is a Riemannian metric on $\mathbb{H}_{n,m}$ which is invariant under the action (1.2) of G^J. In fact, $ds^2_{n,m;A,B}$ is a Kähler metric of $\mathbb{H}_{n,m}$. The author [29] proved that for any two positive real numbers A and B, the following differential operator

$$\Delta_{n,m;A,B} = \frac{4}{A}\, \mathbb{M}_2 + \frac{4}{B} \mathbb{M}_1 \tag{4.6}$$

is the Laplacian of the G^J-invariant Riemannian metric $ds^2_{n,m;A,B}$.

We set, for an integer k with $1 \leq k \leq m$,

$$\frac{\partial}{\partial Z_k} = {}^t\!\left(\frac{\partial}{\partial z_{1k}}, \cdots, \frac{\partial}{\partial z_{nk}} \right)$$

and

$$Y_{+,k} := \frac{\partial}{\partial Z_k}, \quad Y_{-,k} := \frac{\partial}{\partial \overline{Z}_k} Y.$$

We define

$$\begin{aligned} Y_+ &:= \frac{\partial}{\partial Z}, \qquad Y_- := {}^t\!\left(\frac{\partial}{\partial \overline{Z}}\right) Y, \\ X_+ &:= 2\,i\,\frac{\partial}{\partial \Omega} + i\,Y^{-1}\,{}^tV \frac{\partial}{\partial Z} + {}^t\!\left(i\,Y^{-1}\,{}^tV\,{}^t\!\left(\frac{\partial}{\partial Z}\right)\right), \\ X_- &:= {}^tY_-\,{}^t\!\left(Y\overline{Y}_+\right), \\ \widetilde{K} &:= 2\,i\,Y\,\frac{\partial}{\partial \Omega} + i\,{}^tV\,{}^t\!\left(\frac{\partial}{\partial Z}\right) + i\,{}^t\!\left(Y^{-1}\,{}^tV\,{}^t\!\left(\frac{\partial}{\partial Z}\right) Y\right), \end{aligned}$$

and

$$\widetilde{\Lambda} := 2\,i\,Y\,\frac{\partial}{\partial \overline{\Omega}} + i\,{}^tV\,{}^t\!\left(\frac{\partial}{\partial \overline{Z}}\right) + i\,{}^t\!\left(Y^{-1}\,{}^tV\,{}^t\!\left(\frac{\partial}{\partial \overline{Z}}\right) Y\right).$$

Following H. Maass [15] (cf. (2.18)–(2.20)), we put

$$\widetilde{A}^{(1)} = \widetilde{\Lambda}\widetilde{K} + \frac{n+1}{2}\widetilde{K}$$

and define $\widetilde{A}^{(j)}$ $(j = 2, 3, \cdots, n)$ recursively by

$$\begin{aligned} \widetilde{A}^{(j)} = {} & \widetilde{A}^{(1)}\widetilde{A}^{(j-1)} - \frac{n+1}{2}\,\widetilde{\Lambda}\,\widetilde{A}^{(j-1)} + \frac{1}{2}\,\widetilde{\Lambda}\,\mathrm{tr}\!\left(\widetilde{A}^{(j-1)}\right) \\ & + \frac{1}{2}\left(\Omega - \overline{\Omega}\right)\,{}^t\!\left\{\left(\Omega - \overline{\Omega}\right)^{-1}\,{}^t\!\left({}^t\widetilde{\Lambda}\,{}^t\widetilde{A}^{(j-1)}\right)\right\}. \end{aligned}$$

For any positive integers j, k, l with $1 \le j \le n,\ 1 \le k, l \le m$, we define

$$\begin{aligned} \widetilde{H}_j :&= \mathrm{tr}\!\left(\widetilde{A}^{(j)}\right), \qquad T_{k,l} := \mathrm{tr}\left({}^tY_{-,k}\,{}^tY_{+,l}\,\widetilde{A}^{(j)}\right), \\ U_{k,l} :&= \mathrm{tr}\left({}^tY_{-,k}\,Y_{-,l}\,X_+\right), \qquad V_{k,l} := \mathrm{tr}\left(Y_{+,k}\,{}^tY_{+,l}\,X_-\right). \end{aligned}$$

J. Yang and L. Yin [32] showed that $\widetilde{H}_j$, $T_{k,l}$, $U_{k,l}$ and $V_{k,l}$ are invariant under the action (1.2) of G^J.

5 The Partial Cayley Transform

In this section we discuss a notion of the partial Cayley transform and give examples of explicit G^J-invariant differential operators on the Siegel-Jacobi disk.

Let

$$\mathbb{D}_n = \left\{ W \in \mathbb{C}^{(n,n)} \mid W = {}^tW,\ I_n - \overline{W}W > 0 \right\}$$

be the generalized unit disk.

For brevity, we write $\mathbb{D}_{n,m} := \mathbb{D}_n \times \mathbb{C}^{(m,n)}$. This homogeneous space $\mathbb{D}_{n,m}$ is called the *Siegel-Jacobi disk* of degree n and index m. For a coordinate $(W,\eta) \in \mathbb{D}_{n,m}$ with $W = (w_{\mu\nu}) \in \mathbb{D}_n$ and $\eta = (\eta_{kl}) \in \mathbb{C}^{(m,n)}$, we put

$$\begin{aligned} dW &= (dw_{\mu\nu}), \qquad d\overline{W} = (d\overline{w}_{\mu\nu}), \\ d\eta &= (d\eta_{kl}), \qquad d\overline{\eta} = (d\overline{\eta}_{kl}) \end{aligned}$$

and

$$\frac{\partial}{\partial W} = \left(\frac{1+\delta_{\mu\nu}}{2} \frac{\partial}{\partial w_{\mu\nu}} \right), \quad \frac{\partial}{\partial \overline{W}} = \left(\frac{1+\delta_{\mu\nu}}{2} \frac{\partial}{\partial \overline{w}_{\mu\nu}} \right),$$

$$\frac{\partial}{\partial \eta} = \begin{pmatrix} \frac{\partial}{\partial \eta_{11}} & \cdots & \frac{\partial}{\partial \eta_{m1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial \eta_{1n}} & \cdots & \frac{\partial}{\partial \eta_{mn}} \end{pmatrix}, \quad \frac{\partial}{\partial \overline{\eta}} = \begin{pmatrix} \frac{\partial}{\partial \overline{\eta}_{11}} & \cdots & \frac{\partial}{\partial \overline{\eta}_{m1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial \overline{\eta}_{1n}} & \cdots & \frac{\partial}{\partial \overline{\eta}_{mn}} \end{pmatrix}.$$

We can identify an element $g = (M,(\lambda,\mu;\kappa))$ of G^J, $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R})$ with the element

$$\begin{pmatrix} A & 0 & B & A\,{}^t\mu - B\,{}^t\lambda \\ \lambda & I_m & \mu & \kappa \\ C & 0 & D & C\,{}^t\mu - D\,{}^t\lambda \\ 0 & 0 & 0 & I_m \end{pmatrix}$$

of $Sp(m+n,\mathbb{R})$.

We set

$$T_* = \frac{1}{\sqrt{2}} \begin{pmatrix} I_{m+n} & I_{m+n} \\ iI_{m+n} & -iI_{m+n} \end{pmatrix}.$$

We now consider the group G^J_* defined by

$$G^J_* := T_*^{-1} G^J T_*.$$

If $g = (M,(\lambda,\mu;\kappa)) \in G^J$ with $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R})$, then $T_*^{-1} g T_*$ is given by

$$T_*^{-1} g T_* = \begin{pmatrix} P_* & Q_* \\ \overline{Q}_* & \overline{P}_* \end{pmatrix}, \tag{5.1}$$

where

$$P_* = \begin{pmatrix} P & \frac{1}{2}\left\{ Q\,{}^t(\lambda+i\mu) - P\,{}^t(\lambda-i\mu) \right\} \\ \frac{1}{2}(\lambda+i\mu) & I_h + i\frac{\kappa}{2} \end{pmatrix},$$

$$Q_* = \begin{pmatrix} Q & \frac{1}{2}\left\{ P\,{}^t(\lambda-i\mu) - Q\,{}^t(\lambda+i\mu) \right\} \\ \frac{1}{2}(\lambda-i\mu) & -i\frac{\kappa}{2} \end{pmatrix},$$

and P, Q are given by the formulas

$$P = \frac{1}{2}\,\{(A+D) + \,i\,(B-C)\} \tag{5.2}$$

and

$$Q = \frac{1}{2}\,\{(A-D) - \,i\,(B+C)\}\,. \tag{5.3}$$

From now on, we write

$$\left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, \left(\frac{1}{2}(\lambda + i\mu),\; \frac{1}{2}(\lambda - i\mu);\; -i\frac{\kappa}{2} \right) \right) := \begin{pmatrix} P_* & Q_* \\ \overline{Q}_* & \overline{P}_* \end{pmatrix}.$$

In other words, we have the relation

$$T_*^{-1} \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, (\lambda, \mu; \kappa) \right) T_* = \left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, \left(\frac{1}{2}(\lambda + i\mu),\; \frac{1}{2}(\lambda - i\mu);\; -i\frac{\kappa}{2} \right) \right).$$

Let

$$H_{\mathbb{C}}^{(n,m)} := \left\{ (\xi, \eta\,; \zeta) \mid \xi, \eta \in \mathbb{C}^{(m,n)},\ \zeta \in \mathbb{C}^{(m,m)},\ \zeta + \eta\,{}^t\xi \text{ symmetric} \right\}$$

be the complex Heisenberg group endowed with the following multiplication

$$(\xi, \eta\,; \zeta) \circ (\xi', \eta'; \zeta') := (\xi + \xi', \eta + \eta'\,; \zeta + \zeta' + \xi\,{}^t\eta' - \eta\,{}^t\xi')).$$

We define the semidirect product

$$SL(2n, \mathbb{C}) \ltimes H_{\mathbb{C}}^{(n,m)}$$

endowed with the following multiplication

$$\begin{aligned} &\left(\begin{pmatrix} P & Q \\ R & S \end{pmatrix}, (\xi, \eta\,; \zeta) \right) \cdot \left(\begin{pmatrix} P' & Q' \\ R' & S' \end{pmatrix}, (\xi', \eta'; \zeta') \right) \\ = &\left(\begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} P' & Q' \\ R' & S' \end{pmatrix}, (\tilde{\xi} + \xi', \tilde{\eta} + \eta'; \zeta + \zeta' + \tilde{\xi}\,{}^t\eta' - \tilde{\eta}\,{}^t\xi') \right), \end{aligned}$$

where $\tilde{\xi} = \xi P' + \eta R'$ and $\tilde{\eta} = \xi Q' + \eta S'$.

If we identify $H_{\mathbb{R}}^{(n,m)}$ with the subgroup

$$\left\{ (\xi, \overline{\xi}; i\kappa) \mid \xi \in \mathbb{C}^{(m,n)},\ \kappa \in \mathbb{R}^{(m,m)} \right\}$$

of $H_{\mathbb{C}}^{(n,m)}$, we have the following inclusion

$$G_*^J \subset SU(n,n) \ltimes H_{\mathbb{R}}^{(n,m)} \subset SL(2n, \mathbb{C}) \ltimes H_{\mathbb{C}}^{(n,m)}.$$

We define the mapping $\Theta : G^J \longrightarrow G^J_*$ by

$$\Theta\left(\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, (\lambda, \mu; \kappa)\right)\right) = \left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, \left(\frac{1}{2}(\lambda + i\mu),\ \frac{1}{2}(\lambda - i\mu);\ -i\frac{\kappa}{2}\right)\right), \quad (5.4)$$

where P and Q are given by (5.2) and (5.3). We can see that if $g_1, g_2 \in G^J$, then $\Theta(g_1 g_2) = \Theta(g_1)\Theta(g_2)$.

According to [26, p. 250], G^J_* is of the Harish-Chandra type (cf. [17, p. 118]). Let

$$g_* = \left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, (\lambda, \mu;\ \kappa)\right)$$

be an element of G^J_*. Since the Harish-Chandra decomposition of an element $\begin{pmatrix} P & Q \\ R & S \end{pmatrix}$ in $SU(n,n)$ is given by

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} I_n & QS^{-1} \\ 0 & I_n \end{pmatrix} \begin{pmatrix} P - QS^{-1}R & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} I_n & 0 \\ S^{-1}R & I_n \end{pmatrix},$$

the P^+_*-component of the following element

$$g_* \cdot \left(\begin{pmatrix} I_n & W \\ 0 & I_n \end{pmatrix}, (0, \eta; 0)\right), \quad W \in \mathbb{D}_n$$

of $SL(2n, \mathbb{C}) \ltimes H^{(n,m)}_{\mathbb{C}}$ is given by

$$\left(\begin{pmatrix} I_n & (PW + Q)(\overline{Q}W + \overline{P})^{-1} \\ 0 & I_n \end{pmatrix}, \left(0,\ (\eta + \lambda W + \mu)(\overline{Q}W + \overline{P})^{-1};0\right)\right). \quad (5.5)$$

We can identify $\mathbb{D}_{n,m}$ with the subset

$$\left\{ \left(\begin{pmatrix} I_n & W \\ 0 & I_n \end{pmatrix}, (0, \eta; 0)\right) \,\middle|\, W \in \mathbb{D}_n,\ \eta \in \mathbb{C}^{(m,n)} \right\}$$

of the complexification of G^J_*. Indeed, $\mathbb{D}_{n,m}$ is embedded into P^+_* given by

$$P^+_* = \left\{ \left(\begin{pmatrix} I_n & W \\ 0 & I_n \end{pmatrix}, (0, \eta; 0)\right) \,\middle|\, W = {}^tW \in \mathbb{C}^{(n,n)},\ \eta \in \mathbb{C}^{(m,n)} \right\}.$$

This is a generalization of the Harish-Chandra embedding (cf. [17, p. 119]). Then we get the *natural transitive action* of G^J_* on $\mathbb{D}_{n,m}$ defined by

$$\begin{aligned} &\left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, (\xi, \overline{\xi};\ i\kappa)\right) \cdot (W, \eta) \qquad (5.6) \\ &= \left((PW + Q)(\overline{Q}W + \overline{P})^{-1}, (\eta + \xi W + \overline{\xi})(\overline{Q}W + \overline{P})^{-1}\right), \end{aligned}$$

where $\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \in G_*,\ \xi \in \mathbb{C}^{(m,n)},\ \kappa \in \mathbb{R}^{(m,m)}$ and $(W,\eta) \in \mathbb{D}_{n,m}$.

The author [30] proved that the action (1.2) of G^J on $\mathbb{H}_{n,m}$ is compatible with the action (5.6) of G_*^J on $\mathbb{D}_{n,m}$ through the *partial Cayley transform* $\Psi : \mathbb{D}_{n,m} \longrightarrow \mathbb{H}_{n,m}$ defined by

$$\Psi(W,\eta) := \Big(i(I_n + W)(I_n - W)^{-1},\, 2\,i\,\eta\,(I_n - W)^{-1} \Big).$$

In other words, if $g_0 \in G^J$ and $(W,\eta) \in \mathbb{D}_{n,m}$,

$$g_0 \cdot \Psi(W,\eta) = \Psi(g_* \cdot (W,\eta)),$$

where $g_* = T_*^{-1} g_0 T_*$. Ψ is a biholomorphic mapping of $\mathbb{D}_{n,m}$ onto $\mathbb{H}_{n,m}$ which gives the partially bounded realization of $\mathbb{H}_{n,m}$ by $\mathbb{D}_{n,m}$. The inverse of Ψ is

$$\Psi^{-1}(\Omega, Z) = \Big((\Omega - iI_n)(\Omega + iI_n)^{-1},\, Z(\Omega + iI_n)^{-1} \Big).$$

The author [31] proved that for any two positive real numbers A and B, the following metric $d\tilde{s}^2_{n,m;A,B}$ defined by

$$\begin{aligned}
ds^2_{\mathbb{D}_{n,m};A,B} = {} & 4\,A\,\mathrm{tr}\Big((I_n - W\overline{W})^{-1} dW (I_n - \overline{W}W)^{-1} d\overline{W} \Big) \\
& + 4\,B \Big\{ \mathrm{tr}\Big((I_n - W\overline{W})^{-1}\, {}^t(d\eta)\,\beta \Big) \\
& \quad + \mathrm{tr}\Big((\eta\overline{W} - \overline{\eta})(I_n - W\overline{W})^{-1} dW (I_n - \overline{W}W)^{-1}\, {}^t(d\overline{\eta}) \Big) \\
& \quad + \mathrm{tr}\Big((\overline{\eta}W - \eta)(I_n - \overline{W}W)^{-1} d\overline{W} (I_n - W\overline{W})^{-1}\, {}^t(d\eta) \Big) \\
& \quad - \mathrm{tr}\Big((I_n - W\overline{W})^{-1}\, {}^t\eta\,\eta\,(I_n - \overline{W}W)^{-1}\overline{W} dW (I_n - \overline{W}W)^{-1} d\overline{W} \Big) \\
& \quad - \mathrm{tr}\Big(W (I_n - \overline{W}W)^{-1}\, {}^t\overline{\eta}\,\overline{\eta}\,(I_n - W\overline{W})^{-1} dW (I_n - \overline{W}W)^{-1} d\overline{W} \Big) \\
& \quad + \mathrm{tr}\Big((I_n - W\overline{W})^{-1\,t}\eta\,\overline{\eta}\,(I_n - W\overline{W})^{-1} dW (I_n - \overline{W}W)^{-1} d\overline{W} \Big) \\
& \quad + \mathrm{tr}\Big((I_n - \overline{W})^{-1}\, {}^t\overline{\eta}\,\eta\,\overline{W}\,(I_n - W\overline{W})^{-1} dW (I_n - \overline{W}W)^{-1} d\overline{W} \Big) \\
& \quad + \mathrm{tr}\Big((I_n - \overline{W})^{-1}(I_n - W)(I_n - \overline{W}W)^{-1}\, {}^t\overline{\eta}\,\eta\,(I_n - \overline{W}W)^{-1} \\
& \qquad\quad \times (I_n - \overline{W})(I_n - W)^{-1} dW (I_n - \overline{W}W)^{-1} d\overline{W} \Big) \\
& \quad - \mathrm{tr}\Big((I_n - W\overline{W})^{-1}(I_n - W)(I_n - \overline{W})^{-1}\, {}^t\overline{\eta}\,\eta\,(I_n - W)^{-1} \\
& \qquad\quad \times dW (I_n - \overline{W}W)^{-1} d\overline{W} \Big) \Big\}
\end{aligned}$$

is a Riemannian metric on $\mathbb{D}_{n,m}$ which is invariant under the action (5.6) of the Jacobi group G^J_*.

The author [31] proved that the following differential operators $\mathbb{S}_1$ and $\mathbb{S}_2$ on $\mathbb{D}_{n,m}$ defined by

$$\mathbb{S}_1 = \sigma\left((I_n - \overline{W}W)\frac{\partial}{\partial\eta}\,{}^t\!\left(\frac{\partial}{\partial\overline{\eta}}\right)\right)$$

and

$$\begin{aligned}
\mathbb{S}_2 = \ & \mathrm{tr}\left((I_n - W\overline{W})\,{}^t\!\left((I_n - W\overline{W})\frac{\partial}{\partial\overline{W}}\right)\frac{\partial}{\partial W}\right) \\
& + \mathrm{tr}\left({}^t(\eta - \overline{\eta}\,W)\,{}^t\!\left(\frac{\partial}{\partial\overline{\eta}}\right)(I_n - \overline{W}W)\frac{\partial}{\partial W}\right) \\
& + \mathrm{tr}\left((\overline{\eta} - \eta\,\overline{W})\,{}^t\!\left((I_n - W\overline{W})\frac{\partial}{\partial\overline{W}}\right)\frac{\partial}{\partial\eta}\right) \\
& - \mathrm{tr}\left(\eta\overline{W}(I_n - W\overline{W})^{-1}\,{}^t\eta\,{}^t\!\left(\frac{\partial}{\partial\overline{\eta}}\right)(I_n - \overline{W}W)\frac{\partial}{\partial\eta}\right) \\
& - \mathrm{tr}\left(\overline{\eta}W(I_n - \overline{W}W)^{-1}\,{}^t\overline{\eta}\,{}^t\!\left(\frac{\partial}{\partial\overline{\eta}}\right)(I_n - \overline{W}W)\frac{\partial}{\partial\eta}\right) \\
& + \mathrm{tr}\left(\overline{\eta}(I_n - W\overline{W})^{-1}\,{}^t\eta\,{}^t\!\left(\frac{\partial}{\partial\overline{\eta}}\right)(I_n - \overline{W}W)\frac{\partial}{\partial\eta}\right) \\
& + \mathrm{tr}\left(\eta\,\overline{W}W(I_n - \overline{W}W)^{-1}\,{}^t\overline{\eta}\,{}^t\!\left(\frac{\partial}{\partial\overline{\eta}}\right)(I_n - \overline{W}W)\frac{\partial}{\partial\eta}\right)
\end{aligned}$$

are invariant under the action (5.6) of G^J_*. The author also proved that

$$\Delta_{\mathbb{D}_{n,m};A,B} := \frac{1}{A}\,\mathbb{S}_2 + \frac{1}{B}\,\mathbb{S}_1 \tag{5.7}$$

is the Laplacian of the invariant metric $ds^2_{\mathbb{D}_{n,m};A,B}$ on $\mathbb{D}_{n,m}$ (cf. [31]).

The authors [13] proved that the following differential operator on $\mathbb{D}_{n,m}$ defined by

$$\mathbb{S}_3 = \det(I_n - \overline{W}W)\,\det\left(\frac{\partial}{\partial\eta}\,{}^t\!\left(\frac{\partial}{\partial\overline{\eta}}\right)\right)$$

is invariant under the action (5.6) of G^J_* on $\mathbb{D}_{n,m}$. Furthermore the authors [13] proved that the following matrix-valued differential operator on $\mathbb{D}_{n,m}$ defined by

$$\mathbb{J} := {}^t\!\left(\frac{\partial}{\partial\overline{\eta}}\right)(I_n - \overline{W}W)\frac{\partial}{\partial\eta}$$

and each (k,l)-entry $\mathbb{J}_{kl}$ of $\mathbb{J}$ given by

$$\mathbb{J}_{kl} = \sum_{i,j=1}^{n} \left(\delta_{ij} - \sum_{r=1}^{n} \overline{w}_{ir}\, w_{jr} \right) \frac{\partial^2}{\partial \overline{\eta}_{ki} \partial \eta_{lj}}, \quad 1 \leq k, l \leq m$$

are invariant under the action (5.6) of G_*^J on $\mathbb{D}_{n,m}$.

$$\mathbb{S}_* = [\mathbb{S}_1, \mathbb{S}_2] = \mathbb{S}_1 \mathbb{S}_2 - \mathbb{S}_2 \mathbb{S}_1$$

is an invariant differential operator of degree three on $\mathbb{D}_{n,m}$ and

$$\mathbb{Q}_{kl} = [\mathbb{S}_3, \mathbb{J}_{kl}] = \mathbb{S}_3 \mathbb{J}_{kl} - \mathbb{J}_{kl} \mathbb{S}_3, \quad 1 \leq k, l \leq m$$

is an invariant differential operator of degree $2n+1$ on $\mathbb{D}_{n,m}$.

Indeed it is very complicated and difficult at this moment to express the generators of the algebra of all G_*^J-invariant differential operators on $\mathbb{D}_{n,m}$ explicitly.

6 Invariant Differential Operators on the Siegel-Jacobi Space of Lowest Dimension

We consider the case $n = m = 1$. For a coordinate (w, ξ) in $T_{1,1} = \mathbb{C} \times \mathbb{C}$, we write $w = r + i\, s,\ \xi = \zeta + i\, \eta \in \mathbb{C},\ r, s, \zeta, \eta$ real. The author [27] proved that the algebra $\mathrm{Pol}_{1,1}^{U(1)}$ is generated by

$$\begin{aligned}
q(w,\xi) &= \frac{1}{4}\, w\, \overline{w} = \frac{1}{4}\big(r^2 + s^2\big), \\
\alpha(w,\xi) &= \xi\, \overline{\xi} = \zeta^2 + \eta^2, \\
\phi(w,\xi) &= \frac{1}{2}\, \mathrm{Re}\, \big(\xi^2 \overline{w}\big) = \frac{1}{2}\, r\big(\zeta^2 - \eta^2\big) + s\, \zeta \eta, \\
\psi(w,\xi) &= \frac{1}{2}\, \mathrm{Im}\, \big(\xi^2 \overline{w}\big) = \frac{1}{2}\, s\big(\eta^2 - \zeta^2\big) + r\, \zeta \eta.
\end{aligned}$$

In [27], using Formula (3.6) the author calculated explicitly the images

$$D_1 = \Theta_{1,1}(q), \quad D_2 = \Theta_{1,1}(\alpha), \quad D_3 = \Theta_{1,1}(\phi) \quad \text{and} \quad D_4 = \Theta_{1,1}(\psi)$$

of q, ξ, ϕ and ψ under the Helgason map $\Theta_{1,1}$. We can show that the algebra $\mathbb{D}(\mathbb{H}_{1,1})$ is generated by the following differential operators

$$\begin{aligned} D_1 =& y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + v^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) \\ & + 2\, y\, v \left(\frac{\partial^2}{\partial x \partial u} + \frac{\partial^2}{\partial y \partial v} \right), \end{aligned}$$

$$D_2 = y \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right),$$

$$\begin{aligned} D_3 =& y^2 \frac{\partial}{\partial y} \left(\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} \right) - 2y^2 \frac{\partial^3}{\partial x \partial u \partial v} \\ & - \left(v \frac{\partial}{\partial v} + 1 \right) D_2 \end{aligned}$$

and

$$\begin{aligned} D_4 =& y^2 \frac{\partial}{\partial x} \left(\frac{\partial^2}{\partial v^2} - \frac{\partial^2}{\partial u^2} \right) - 2\, y^2 \frac{\partial^3}{\partial y \partial u \partial v} \\ & - v \frac{\partial}{\partial u} D_2, \end{aligned}$$

where $\tau = x + iy$ and $z = u + iv$ with real variables x, y, u, v. Moreover, we have

$$\begin{aligned} D_1 D_2 - D_2 D_1 \;=&\; 2\, y^2 \frac{\partial}{\partial y} \left(\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} \right) \\ & - 4\, y^2 \frac{\partial^3}{\partial x \partial u \partial v} - 2 \left(v \frac{\partial}{\partial v} D_2 + D_2 \right). \end{aligned}$$

In particular, the algebra $\mathbb{D}(\mathbb{H}_{1,1})$ is not commutative. We refer to [2, 6, 27] for more detail.

Recently the authors [13] proved the following results.

Theorem 6.3. *We have the following relation*

$$\phi^2 + \psi^2 = q\, \alpha^2.$$

This relation exhausts all the relations among the generators q, α, ϕ *and* ψ *of* $\mathrm{Pol}_{1,1}^{U(1)}$.

Theorem 6.4. *We have the following relations*

(a) $[D_1, D_2] = 2D_3$

(b) $[D_1, D_3] = 2D_1 D_2 - 2D_3$

(c) $[D_2, D_3] = -D_2^2$

(*d*) $[D_4, D_1] = 0$

(*e*) $[D_4, D_2] = 0$

(*f*) $[D_4, D_3] = 0$

(*g*) $D_3^2 + D_4^2 = D_2 D_1 D_2$

These seven relations exhaust all the relations among the generators D_1, D_2, D_3 *and* D_4 *of* $\mathbb{D}(\mathbb{H}_{1,1})$.

Theorem 6.5. *The action of* $U(1)$ *on* $\mathrm{Pol}_{1,1}^{U(1)}$ *is not multiplicity-free.*

Finally we see that for the case when $n = m = 1$, the eight problems proposed in Section 3 are completely solved.

Remark 1. According to Theorem 6.4, we see that D_4 is a generator of the center of $\mathbb{D}(\mathbb{H}_{1,1})$. We observe that the Lapalcian

$$\Delta_{1,1;A,B} = \frac{4}{A}\, D_1 \,+\, \frac{4}{B}\, D_2 \qquad \text{(see (4.6))}$$

of $(\mathbb{H}_{1,1}, ds^2_{1,1;A,B})$ does not belong to the center of $\mathbb{D}(\mathbb{H}_{1,1})$.

Remark 2. When $n = 1$ and m is an arbitrary integer, Conley and Raum [6] found the $2m^2 + m + 1$ explicit generators of $\mathbb{D}(\mathbb{H}_{1,m})$ and the explicit one generator of the center of $\mathbb{D}(\mathbb{H}_{1,m})$. They also found the generators of the center of the universal enveloping algebra of $\mathfrak{U}(\mathfrak{g}^J)$ of the Jacobi Lie algebra $\mathfrak{g}^J$. The number of generators of the center of $\mathfrak{U}(\mathfrak{g}^J)$ is $1 + \frac{m(m+1)}{2}$.

7 Remarks on Maass-Jacobi Forms

Using G^J-invariant differential operators on the Siegel-Jacobi space, we introduce a notion of Maass-Jacobi forms.

Definition 1. Let

$$\Gamma_{n,m} := Sp(n,\mathbb{Z}) \ltimes H_{\mathbb{Z}}^{(n,m)}$$

be the discrete subgroup of G^J, where

$$H_{\mathbb{Z}}^{(n,m)} = \left\{ (\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)} \mid \lambda,\mu,\kappa \text{ are integral } \right\}.$$

A smooth function $f : \mathbb{H}_{n,m} \longrightarrow \mathbb{C}$ is called a Maass-Jacobi form on $\mathbb{H}_{n,m}$ if f satisfies the following conditions (MJ1)-(MJ3):

(MJ1) f is invariant under $\Gamma_{n,m}$.

(MJ2) f is an eigenfunction of the Laplacian $\Delta_{n,m;A,B}$ (cf. Formula (4.6)).
(MJ3) f has a polynomial growth, that is, there exist a constant $C > 0$ and a positive integer N such that

$$|f(X + i\,Y, Z)| \le C\,|p(Y)|^N \quad \text{as } \det Y \longrightarrow \infty,$$

where $p(Y)$ is a polynomial in $Y = (y_{ij})$.

Remark 3. Let $\mathbb{D}_*$ be a commutative subalgebra of $\mathbb{D}(\mathbb{H}_{n,m})$ containing the Laplacian $\Delta_{n,m;A,B}$. We say that a smooth function $f : \mathbb{H}_{n,m} \longrightarrow \mathbb{C}$ is a Maass-Jacobi form with respect to $\mathbb{D}_*$ if f satisfies the conditions $(MJ1)$, $(MJ2)_*$ and $(MJ3)$: the condition $(MJ2)_*$ is given by

$(MJ2)_*$ f is an eigenfunction of any invariant differential operator in $\mathbb{D}_*$.

Remark 4. Erik Balslev [1] developed the spectral theory of $\Delta_{1,1;1,1}$ on $\mathbb{H}_{1,1}$ to prove that the set of all eigenvalues of $\Delta_{1,1;1,1}$ satisfies the Weyl law.

It is natural to propose the following problems.

Problem A : Find all the eigenfunctions of $\Delta_{n,m;A,B}$.

Problem B : Construct Maass-Jacobi forms.

If we find a *nice* eigenfunction ϕ of the Laplacian $\Delta_{n,m;A,B}$, we can construct a Maass-Jacobi form f_ϕ on $\mathbb{H}_{n,m}$ in the usual way defined by

$$f_\phi(\Omega, Z) := \sum_{\gamma \in \Gamma^\infty_{n,m} \backslash \Gamma_{n,m}} \phi\big(\gamma \cdot (\Omega, Z)\big),$$

where

$$\Gamma^\infty_{n,m} = \left\{ \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, (\lambda, \mu; \kappa) \right) \in \Gamma_{n,m} \;\Big|\; C = 0 \right\}$$

is a subgroup of $\Gamma_{n,m}$.

We consider the simple case when $n = m = 1$ and $A = B = 1$. A metric $ds^2_{1,1;1,1}$ on $\mathbb{H}_{1,1}$ given by

$$\begin{aligned} ds^2_{1,1;1,1} = &\ \frac{y + v^2}{y^3}\,(\,dx^2 + dy^2\,) + \frac{1}{y}\,(\,du^2 + dv^2\,) \\ &- \frac{2v}{y^2}\,(\,dx\,du + dy\,dv\,) \end{aligned}$$

is a G^J-invariant Kähler metric on $\mathbb{H}_{1,1}$. Its Laplacian $\Delta_{1,1;1,1}$ is given by

$$\begin{aligned}\Delta_{1,1;1,1} \,&= y^2\,\left(\frac{\partial^2}{\partial x^2}\,+\,\frac{\partial^2}{\partial y^2}\right)\\ &\quad+\,(\,y\,+\,v^2\,)\left(\frac{\partial^2}{\partial u^2}\,+\,\frac{\partial^2}{\partial v^2}\right)\\ &\quad+\,2\,y\,v\,\left(\frac{\partial^2}{\partial x\partial u}\,+\,\frac{\partial^2}{\partial y\partial v}\right).\end{aligned}$$

We provide some examples of eigenfunctions of $\Delta_{1,1;1,1}$.

(a) $h(x,y)=y^{\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|a|y)\,e^{2\pi iax}$ $(s\in\mathbb{C},\,a\neq 0\,)$ with eigenvalue $s(s-1)$. Here

$$K_s(z):=\frac{1}{2}\int_0^\infty \exp\left\{-\frac{z}{2}(t+t^{-1})\right\}\,t^{s-1}\,dt,$$

where $\operatorname{Re} z>0$.

(b) $y^s,\ y^s x,\ y^s u$ $(s\in\mathbb{C})$ with eigenvalue $s(s-1)$.

(c) $y^s v,\ y^s uv,\ y^s xv$ with eigenvalue $s(s+1)$.

(d) $x,\,y,\,u,\,v,\,xv,\,uv$ with eigenvalue 0.

(e) All Maass wave forms.

Let ρ be a rational representation of $GL(n,\mathbb{C})$ on a finite dimensional complex vector space V_ρ. Let $\mathcal{M}\in\mathbb{R}^{(m,m)}$ be a symmetric half-integral semi-positive definite matrix of degree m. Let $C^\infty(\mathbb{H}_{n,m},V_\rho)$ be the algebra of all C^∞ functions on $\mathbb{H}_{n,m}$ with values in V_ρ. We define the $|_{\rho,\mathcal{M}}$-slash action of G^J on $C^\infty(\mathbb{H}_{n,m},V_\rho)$ as follows: If $f\in C^\infty(\mathbb{H}_{n,m},V_\rho)$,

$$\begin{aligned}&f|_{\rho,\mathcal{M}}[(M,(\lambda,\mu;\kappa))](\Omega,Z)\\ :=\;& e^{-2\pi i\,\mathrm{tr}(\mathcal{M}[Z+\lambda\Omega+\mu](C\Omega+D)^{-1}C)}\cdot e^{2\pi i\,\mathrm{tr}(\mathcal{M}(\lambda\Omega\,{}^t\lambda+2\lambda\,{}^tZ+\kappa+\mu\,{}^t\lambda))}\\ &\times\rho(C\Omega+D)^{-1}f(M\cdot\Omega,(Z+\lambda\Omega+\mu)(C\Omega+D)^{-1}),\end{aligned}$$

where $\begin{pmatrix}A&B\\C&D\end{pmatrix}\in Sp(n,\mathbb{R})$ and $(\lambda,\mu;\kappa)\in H_{\mathbb{R}}^{(n,m)}$. We recall the Siegel's notation $\alpha[\beta]={}^t\beta\alpha\beta$ for suitable matrices α and β. We define $\mathbb{D}_{\rho,\mathcal{M}}$ to be the algebra of all differential operators D on $\mathbb{H}_{n,m}$ satisfying the following condition

$$(Df)|_{\rho,\mathcal{M}}[g]=\,D(f|_{\rho,\mathcal{M}}[g])$$

for all $f\in C^\infty(\mathbb{H}_{n,m},V_\rho)$ and for all $g\in G^J$. We denote by $\mathcal{Z}_{\rho,\mathcal{M}}$ the center of $\mathbb{D}_{\rho,\mathcal{M}}$.

We define another notion of Maass-Jacobi forms as follows.

Definition 2. A vector-valued smooth function $\phi : \mathbb{H}_{n,m} \longrightarrow V_\rho$ is called a Maass-Jacobi form on $\mathbb{H}_{n,m}$ of type ρ and index $\mathcal{M}$ if it satisfies the following conditions $(MJ1)_{\rho,\mathcal{M}},\ (MJ2)_{\rho,\mathcal{M}}$ and $(MJ3)_{\rho,\mathcal{M}}$:

$(MJ1)_{\rho,\mathcal{M}}$ $\phi|_{\rho,\mathcal{M}}[\gamma] = \phi$ for all $\gamma \in \Gamma_{n,m}$.

$(MJ2)_{\rho,\mathcal{M}}$ f is an eigenfunction of all differential operators in the center $\mathcal{Z}_{\rho,\mathcal{M}}$ of $\mathbb{D}_{\rho,\mathcal{M}}$.

$(MJ3)_{\rho,\mathcal{M}}$ f has a growth condition

$$\phi(\Omega, Z) = O\Big(e^{a \det Y} \cdot e^{2\pi \mathrm{tr}(\mathcal{M}[V]Y^{-1})}\Big)$$

as $\det Y \longrightarrow \infty$ for some $a > 0$.

The case when $n = 1,\ m = 1$ and $\rho = \det^k (k = 0, 1, 2, \cdots)$ was studied by R. Berndt and R. Schmidt [2], A. Pitale [16] and K. Bringmann and O. Richter [4]. The case when $n = 1,\ m =$arbitrary and $\rho = \det^k (k = 1, 2, \cdots)$ was investigated by C. Conley and M. Raum [6]. In [6] the authors proved that the center $\mathcal{Z}_{\det^k,\mathcal{M}}$ of $\mathbb{D}_{\det^k,\mathcal{M}}$ is the polynomial algebra with one generator $\mathcal{C}^{k,\mathcal{M}}$, the so-called *Casimir* operator which is a $|_{\det^k,\mathcal{M}}$–slash invariant differential operator of degree three for the case when $n = m = 1$ or of degree four for the case when $n = 1,\ m \geq 2$. Bringmann and Richter [4] considered the Poincaré series $\mathcal{P}^{(n,r)}_{k,\mathcal{M},s}$ (the case when $n = m = 1$) that is a *harmonic* Maass-Jacobi form in the sense of Definition 2 and investigated its Fourier expansion and its Fourier coefficients. Here the *harmonicity* of $\mathcal{P}^{(n,r)}_{k,\mathcal{M},s}$ means that $\mathcal{C}^{k,\mathcal{M}}\mathcal{P}^{(n,r)}_{k,\mathcal{M},s} = 0$, i.e., $\mathcal{P}^{(n,r)}_{k,\mathcal{M},s}$ is an eigenfunction of $\mathcal{C}^{k,\mathcal{M}}$ with zero eigenvalue. Conley and Raum [6] generalized the results in [16] and [4] to the case when $n = 1$ and m is arbitrary.

Remark 5. In [3], Bringmann, Conley and Richter proved that the center of the algebra of differential operators invariant under the action of the Jacobi group over a complex quadratic field is generated by two Casimir operators of degree three. They also introduce an analogue of Kohnen's plus space for modular forms of half-integral weight over $K = \mathbb{Q}(i)$, and provide a lift from it to the space of Jacobi forms over K.

References

1. E. Balslev, Spectral theory of the Laplacian on the modular Jacobi group manifold, preprint, Aarhus University (2012).
2. R. Berndt and R. Schmidt, *Elements of the Representation Theory of the Jacobi Group*, Progress in Mathematics, **163**, Birkhäuser, Basel, 1998.
3. K. Bringmann, C. Conley and O. K. Richter, *Jacobi forms over complex quadratic fields via the cubic Casimier operators*, preprint.

4. K. Bringmann and O. K. Richter, *Zagier-type dualities and lifting maps for harmonic Maass-Jacobi forms*, Advances in Math. **225** (2010), 2298-2315.
5. D. Bump and Y. J. Choie, *Derivatives of modular forms of negative weight*, Pure Appl. Math. Q. **2** (2006), no. 1, 111-133.
6. C. Conley and M. Raum, *Harmonic Maass-Jacobi forms of degree 1 with higher rank indices*, arXiv:1012.289/v1 [math.NT] 13 Dec 2010.
7. M. Eichler and D. Zagier, *The Theory of Jacobi Forms*, Progress in Mathematics, **55**, Birkhäuser, Boston, Basel and Stuttgart, 1985.
8. Harish-Chandra, *Representations of a semisimple Lie group on a Banach space. I.,* Trans. Amer. Math. Soc. **75** (1953), 185-243.
9. Harish-Chandra, *The characters of semisimple Lie groups,* Trans. Amer. Math. Soc. **83** (1956), 98-163.
10. S. Helgason, *Differential operators on homogeneous spaces,* Acta Math. **102** (1959), 239-299.
11. S. Helgason, *Groups and geometric analysis,* Academic Press, New York (1984).
12. R. Howe, *Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond,* The Schur lectures (1992) (Tel Aviv), Israel Math. Conf. Proceedings, **vol. 8** (1995), 1-182.
13. M. Itoh, H. Ochiai and J.-H. Yang, Invariant differential operators on Siegel-Jacobi space, preprint (2013).
14. H. Maass, *Die Differentialgleichungen in der Theorie der Siegelschen Modulfunktionen*, Math. Ann. **126** (1953), 44–68.
15. H. Maass, *Siegel modular forms and Dirichlet series,* Lecture Notes in Math., **vol. 216**, Springer-Verlag, Berlin-Heidelberg-New York (1971).
16. A. Pitale, *Jacobi Maass forms*, Abh. Math. Sem. Univ. Hamburg **79** (2009), 87-111.
17. I. Satake, *Algebraic Structures of Symmetric Domains,* Kano Memorial Lectures 4, Iwanami Shoton, Publishers and Princeton University Press (1980).
18. G. Shimura, *Invariant differential operators on hermitian symmetric spaces,* Ann. Math. **132** (1990), 237-272.
19. C. L. Siegel, *Symplectic Geometry,* Amer. J. Math. **65** (1943), 1-86; Academic Press, New York and London (1964); Gesammelte Abhandlungen, no. 41, vol. II, Springer-Verlag (1966), 274-359.
20. C. L. Siegel, *Topics in Complex Function Theory : Abelian Functions and Modular Functions of Several Variables*, vol. III, Wiley-Interscience, 1973.
21. H. Weyl, *The classical groups: Their invariants and representations,* Princeton Univ. Press, Princeton, New Jersey, second edition (1946).
22. J.-H. Yang, *The Siegel-Jacobi Operator*, Abh. Math. Sem. Univ. Hamburg **63** (1993), 135–146.
23. J.-H. Yang, *Singular Jacobi Forms,* Trans. Amer. Math. Soc. **347 (6)** (1995), 2041-2049.
24. J.-H. Yang, *Construction of vector valued modular forms from Jacobi forms,* Canad. J. Math. **47 (6)** (1995), 1329-1339.
25. J.-H. Yang, *A geometrical theory of Jacobi forms of higher degree,* Proceedings of Symposium on Hodge Theory and Algebraic Geometry (edited by Tadao Oda), Sendai, Japan (1996), 125-147 *or* Kyungpook Math. J. **40 (2)** (2000), 209-237.
26. J.-H. Yang, *The Method of Orbits for Real Lie Groups,* Kyungpook Math. J. **42 (2)** (2002), 199-272.
27. J.-H. Yang, *A note on Maass-Jacobi forms,* Kyungpook Math. J. **43 (4)** (2003), 547–566.
28. J.-H. Yang, *A note on a fundamental domain for Siegel-Jacobi space,* Houston J. Math. **32 (3)** (2006), 701–712.
29. J.-H. Yang, *Invariant metrics and Laplacians on Siegel-Jacobi space,* Journal of Number Theory **127** (2007), 83–102.
30. J.-H. Yang, *A partial Cayley transform for Siegel-Jacobi disk,* J. Korean Math. Soc. **45**, No. 3 (2008), 781-794.

31. J.-H. Yang, *Invariant metrics and Laplacians on Siegel-Jacobi disk,* Chinese Annals of Mathematics, Vol. 31B(1), 2010, 85-100.
32. J. Yang and L. Yin, Derivation of Jacobi forms from connections, arXiv:1301.1156v1 [math.NT] 7 Jan 2013.