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Geometry and Arithmetic on the Siegel–Jacobi Space

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To the memory of my teacher, Professor Shoshichi Kobayashi

Abstract. The Siegel–Jacobi space is a non-symmetric homogeneous space which is very important geometrically and arithmetically. In this paper, we discuss the theory of the geometry and the arithmetic of the Siegel–Jacobi space.

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1. Introduction

For a given fixed positive integer n, we let

$$\mathbb{H}_n = \{\, \Omega \in \mathbb{C}^{(n,n)} \mid \Omega = {}^t\Omega, \quad \operatorname{Im}\Omega > 0 \,\}$$

be the Siegel upper half-plane of degree n and let

$$Sp(n,\mathbb{R}) = \{ M \in \mathbb{R}^{(2n,2n)} \mid {}^tMJ_nM = J_n \ \}$$

be the symplectic group of degree n, where $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F for two positive integers k and l, ${}^tM$ denotes the transposed matrix of a matrix M and

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

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Then $Sp(n,\mathbb{R})$ acts on $\mathbb{H}_n$ transitively by

$$M\cdot\Omega=(A\Omega+B)(C\Omega+D)^{-1}, \tag{1.1}$$

where $M=\begin{pmatrix} A & B\\ C & D\end{pmatrix}\in Sp(n,\mathbb{R})$ and $\Omega\in\mathbb{H}_n$. Let

$$\Gamma_n=Sp(n,\mathbb{Z})=\left\{\begin{pmatrix} A & B\\ C & D\end{pmatrix}\in Sp(n,\mathbb{R})\,\Big|\, A,B,C,D \text{ integral}\right\}$$

be the Siegel modular group of degree n. This group acts on $\mathbb{H}_n$ properly discontinuously. C.L. Siegel investigated the geometry of $\mathbb{H}_n$ and automorphic forms on $\mathbb{H}_n$ systematically. Siegel [57] found a fundamental domain $\mathcal{F}_n$ for $\Gamma_n\backslash\mathbb{H}_n$ and described it explicitly. Moreover he calculated the volume of $\mathcal{F}_n$. We also refer to [23], [38], [58] for some details on $\mathcal{F}_n$.

For two positive integers m and n, we consider the Heisenberg group

$$H_{\mathbb{R}}^{(n,m)}=\left\{\,(\lambda,\mu;\kappa)\mid \lambda,\mu\in\mathbb{R}^{(m,n)},\ \kappa\in\mathbb{R}^{(m,m)},\ \kappa+\mu\,{}^t\lambda \text{ symmetric }\right\}$$

endowed with the following multiplication law

$$\big(\lambda,\mu;\kappa\big)\circ\big(\lambda',\mu';\kappa'\big)=\big(\lambda+\lambda',\mu+\mu';\kappa+\kappa'+\lambda\,{}^t\mu'-\mu\,{}^t\lambda'\big)$$

with $\big(\lambda,\mu;\kappa\big),\big(\lambda',\mu';\kappa'\big)\in H_{\mathbb{R}}^{(n,m)}$. We define the *Jacobi group* G^J of degree n and index m that is the semidirect product of $Sp(n,\mathbb{R})$ and $H_{\mathbb{R}}^{(n,m)}$

$$G^J=Sp(n,\mathbb{R})\ltimes H_{\mathbb{R}}^{(n,m)}$$

endowed with the following multiplication law

$$\big(M,(\lambda,\mu;\kappa)\big)\cdot\big(M',(\lambda',\mu';\kappa'\,)\big)=\ \big(MM',(\tilde\lambda+\lambda',\tilde\mu+\mu';\kappa+\kappa'+\tilde\lambda\,{}^t\mu'-\tilde\mu\,{}^t\lambda'\,)\big)$$

with $M,M'\in Sp(n,\mathbb{R}),(\lambda,\mu;\kappa),\,(\lambda',\mu';\kappa')\in H_{\mathbb{R}}^{(n,m)}$ and $(\tilde\lambda,\tilde\mu)=(\lambda,\mu)M'$. Then G^J acts on $\mathbb{H}_n\times\mathbb{C}^{(m,n)}$ transitively by

$$\big(M,(\lambda,\mu;\kappa)\big)\cdot(\Omega,Z)=\Big(M\cdot\Omega,(Z+\lambda\Omega+\mu)(C\Omega+D)^{-1}\Big), \tag{1.2}$$

where $M=\begin{pmatrix} A & B\\ C & D\end{pmatrix}\in Sp(n,\mathbb{R}),\ (\lambda,\mu;\kappa)\in H_{\mathbb{R}}^{(n,m)}$ and $(\Omega,Z)\in\mathbb{H}_n\times\mathbb{C}^{(m,n)}$. We note that the Jacobi group G^J is *not* a reductive Lie group and the homogeneous space $\mathbb{H}_n\times\mathbb{C}^{(m,n)}$ is not a symmetric space. From now on, for brevity we write $\mathbb{H}_{n,m}=\mathbb{H}_n\times\mathbb{C}^{(m,n)}$. The homogeneous space $\mathbb{H}_{n,m}$ is called the *Siegel–Jacobi space* of degree n and index m.

The aim of this paper is to discuss and survey the geometry and the arithmetic of the Siegel–Jacobi space $\mathbb{H}_{n,m}$. This article is organized as follows. In Section 2, we provide Riemannian metrics which are invariant under the action (1.2) of the Jacobi group and their Laplacians. In Section 3, we discuss G^J-invariant differential operators on the Siegel–Jacobi space and give some related results. In Section 4, we describe the partial Cayley transform of the Siegel–Jacobi disk onto the Siegel–Jacobi space which gives a partially bounded realization of the

Siegel–Jacobi space. We provide a compatibility result of a partial Cayley transform. In Section 5, we provide Riemannian metrics on the Siegel–Jacobi disk which is invariant under the action (4.8) of the Jacobi group G^J_* and their Laplacians using the partial Cayley transform. In Section 6, we find a fundamental domain for the Siegel–Jacobi space with respect to the Siegel–Jacobi modular group. In Section 7, we give the canonical automorphic factor for the Jacobi group G^J which is obtained by a geometrical method and review the concept of Jacobi forms. In Section 8, we characterize singular Jacobi forms in terms of a certain differential operator and their weights. In Section 9, we define the notion of the Siegel–Jacobi operator. We give the result about the compatibility with the Hecke–Jacobi operator. In Section 10, we differentiate a given Jacobi form with respect to the toroidal variables by applying a homogeneous pluriharmonic differential operator to a Jacobi form and then obtain a vector-valued modular form of a new weight. As an application, we provide an identity for an Eisenstein series. In Section 11, we discuss the notion of Maass–Jacobi forms. In Section 12, we construct the Schrödinger–Weil representation and give some results on theta sums constructed from the Schrödinger–Weil representation. In Section 13, we give some remarks and propose some open problems about the geometry and the arithmetic of the Siegel–Jacobi space.

Notation. We denote by $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the field of rational numbers, the field of real numbers and the field of complex numbers respectively. We denote by $\mathbb{Z}$ and $\mathbb{Z}^+$ the ring of integers and the set of all positive integers respectively. The symbol ":=" means that the expression on the right is the definition of that on the left. For two positive integers k and l, $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F. For a square matrix $A \in F^{(k,k)}$ of degree k, $\sigma(A)$ denotes the trace of A. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transpose of a matrix M. I_n denotes the identity matrix of degree n. For $A \in F^{(k,l)}$ and $B \in F^{(k,k)}$, we set $B[A] = {}^tABA$. For a complex matrix A, $\overline{A}$ denotes the complex *conjugate* of A. For $A \in \mathbb{C}^{(k,l)}$ and $B \in \mathbb{C}^{(k,k)}$, we use the abbreviation $B\{A\} = {}^t\overline{A}BA$. For a number field F, we denote by $\mathbb{A}_F$ the ring of adeles of F. If $F = \mathbb{Q}$, the subscript will be omitted.

2. Invariant metrics and Laplacians on the Siegel–Jacobi space

For $\Omega = (\omega_{ij}) \in \mathbb{H}_n$, we write $\Omega = X + iY$ with $X = (x_{ij}),\ Y = (y_{ij})$ real. We put $d\Omega = (d\omega_{ij})$ and $d\overline{\Omega} = (d\overline{\omega}_{ij})$. We also put

$$\frac{\partial}{\partial \Omega} = \left(\frac{1+\delta_{ij}}{2}\, \frac{\partial}{\partial \omega_{ij}} \right) \qquad \text{and} \qquad \frac{\partial}{\partial \overline{\Omega}} = \left(\frac{1+\delta_{ij}}{2}\, \frac{\partial}{\partial \overline{\omega}_{ij}} \right).$$

C.L. Siegel [57] introduced the symplectic metric $ds^2_{n;A}$ on $\mathbb{H}_n$ invariant under the action (1.1) of $Sp(n,\mathbb{R})$ that is given by

$$ds^2_{n;A} = A\,\sigma(Y^{-1} d\Omega\, Y^{-1} d\overline{\Omega}), \qquad A > 0 \tag{2.1}$$

and H. Maass [37] proved that its Laplacian is given by

$$\Delta_{n;A} = \frac{4}{A}\,\sigma\left(Y\,{}^t\!\left(Y\frac{\partial}{\partial\overline{\Omega}}\right)\frac{\partial}{\partial\Omega}\right). \tag{2.2}$$

And

$$dv_n(\Omega) = (\det Y)^{-(n+1)} \prod_{1\le i\le j\le n} dx_{ij} \prod_{1\le i\le j\le n} dy_{ij} \tag{2.3}$$

is a $Sp(n,\mathbb{R})$-invariant volume element on $\mathbb{H}_n$ (cf. [59], p. 130).

For a coordinate $(\Omega, Z) \in \mathbb{H}_{n,m}$ with $\Omega = (\omega_{\mu\nu})$ and $Z = (z_{kl})$, we put $d\Omega$, $d\overline{\Omega}$, $\frac{\partial}{\partial\Omega}$, $\frac{\partial}{\partial\overline{\Omega}}$ as before and set

$$Z = U + iV, \qquad U = (u_{kl}), \qquad V = (v_{kl}) \ \text{ real},$$

$$dZ = (dz_{kl}), \qquad d\overline{Z} = (d\overline{z}_{kl}),$$

$$\frac{\partial}{\partial Z} = \begin{pmatrix} \frac{\partial}{\partial z_{11}} & \cdots & \frac{\partial}{\partial z_{m1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial z_{1n}} & \cdots & \frac{\partial}{\partial z_{mn}} \end{pmatrix}, \quad \frac{\partial}{\partial \overline{Z}} = \begin{pmatrix} \frac{\partial}{\partial \overline{z}_{11}} & \cdots & \frac{\partial}{\partial \overline{z}_{m1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial \overline{z}_{1n}} & \cdots & \frac{\partial}{\partial \overline{z}_{mn}} \end{pmatrix}.$$

Yang proved the following theorems in [71].

Theorem 2.1. *For any two positive real numbers A and B,*

$$\begin{aligned} ds^2_{n,m;A,B} = {} & A\,\sigma\Big(Y^{-1}d\Omega\,Y^{-1}d\overline{\Omega}\Big) \\ & + B\Big\{\sigma\Big(Y^{-1}\,{}^tV\,V\,Y^{-1}d\Omega\,Y^{-1}d\overline{\Omega}\Big) + \sigma\Big(Y^{-1}\,{}^t(dZ)\,d\overline{Z}\Big) \\ & - \sigma\Big(V\,Y^{-1}d\Omega\,Y^{-1}\,{}^t(d\overline{Z})\Big) - \sigma\Big(V\,Y^{-1}d\overline{\Omega}\,Y^{-1}\,{}^t(dZ)\Big)\Big\} \end{aligned}$$

is a Riemannian metric on $\mathbb{H}_{n,m}$ which is invariant under the action (1.2) *of G^J. In fact, $ds^2_{n,m;A,B}$ is a Kähler metric of $\mathbb{H}_{n,m}$.*

Proof. See Theorem 1.1 in [71]. □

Theorem 2.2. *The Laplacian $\Delta_{m,m;A,B}$ of the G^J-invariant metric $ds^2_{n,m;A,B}$ is given by*

$$\Delta_{n,m;A,B} = \frac{4}{A}\,\mathbb{M}_1 + \frac{4}{B}\mathbb{M}_2, \tag{2.4}$$

where

$$\begin{aligned} \mathbb{M}_1 = {} & \sigma\left(Y\,{}^t\!\left(Y\frac{\partial}{\partial\overline{\Omega}}\right)\frac{\partial}{\partial\Omega}\right) + \sigma\left(VY^{-1}\,{}^tV\,{}^t\!\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial Z}\right) \\ & + \sigma\left(V\,{}^t\!\left(Y\frac{\partial}{\partial\overline{\Omega}}\right)\frac{\partial}{\partial Z}\right) + \sigma\left({}^tV\,{}^t\!\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial\Omega}\right) \end{aligned}$$

and

$$\mathbb{M}_2 = \sigma\left(Y\frac{\partial}{\partial Z}\,{}^t\!\left(\frac{\partial}{\partial\overline{Z}}\right)\right).$$

Furthermore $\mathbb{M}_1$ *and* $\mathbb{M}_2$ *are differential operators on* $\mathbb{H}_{n,m}$ *invariant under the action* (1.2) *of* G^J.

Proof. See Theorem 1.2 in [71]. □

Remark 2.1. Erik Balslev [2] developed the spectral theory of $\Delta_{1,1;1,1}$ on $\mathbb{H}_{1,1}$ for certain arithmetic subgroups of the Jacobi modular group to prove that the set of all eigenvalues of $\Delta_{1,1;1,1}$ satisfies the Weyl law.

Remark 2.2. The sectional curvature of $(\mathbb{H}_{1,1}, ds^2_{1,1;A,B})$ is $-\frac{3}{A}$ and hence is independent of the parameter B. We refer to [76] for more detail.

Remark 2.3. For an application of the invariant metric $ds^2_{n,m;A,B}$ we refer to [79].

3. Invariant differential operators on the Siegel–Jacobi space

Before we discuss G^J-invariant differential operators on the Siegel–Jacobi space $\mathbb{H}_{n,m}$, we review differential operators on the Siegel upper half-plane $\mathbb{H}_n$ invariant under the action (1.1).

For brevity, we write $G = Sp(n,\mathbb{R})$. The isotropy subgroup K at iI_n for the action (1.1) is a maximal compact subgroup given by

$$K = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \,\Big|\, A\,{}^tA + B\,{}^tB = I_n,\ A\,{}^tB = B\,{}^tA,\ A, B \in \mathbb{R}^{(n,n)} \right\}.$$

Let $\mathfrak{k}$ be the Lie algebra of K. Then the Lie algebra $\mathfrak{g}$ of G has a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where

$$\mathfrak{g} = \left\{ \begin{pmatrix} X_1 & X_2 \\ X_3 & -\,{}^tX_1 \end{pmatrix} \,\Big|\, X_1, X_2, X_3 \in \mathbb{R}^{(n,n)},\ X_2 = {}^tX_2,\ X_3 = {}^tX_3 \right\},$$

$$\mathfrak{k} = \left\{ \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in \mathbb{R}^{(2n,2n)} \,\Big|\, {}^tX + X = 0,\ Y = {}^tY \right\},$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \,\Big|\, X = {}^tX,\ Y = {}^tY,\ X, Y \in \mathbb{R}^{(n,n)} \right\}.$$

The subspace $\mathfrak{p}$ of $\mathfrak{g}$ may be regarded as the tangent space of $\mathbb{H}_n$ at iI_n. The adjoint representation of G on $\mathfrak{g}$ induces the action of K on $\mathfrak{p}$ given by

$$k \cdot Z = kZ\,{}^tk, \quad k \in K,\ Z \in \mathfrak{p}. \tag{3.1}$$

Let T_n be the vector space of $n \times n$ symmetric complex matrices. We let $\Psi : \mathfrak{p} \longrightarrow T_n$ be the map defined by

$$\Psi\left(\begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \right) = X + i\,Y, \quad \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \in \mathfrak{p}. \tag{3.2}$$

We let $\delta : K \longrightarrow U(n)$ be the isomorphism defined by

$$\delta\left(\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \right) = A + i\,B, \quad \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in K, \tag{3.3}$$

where $U(n)$ denotes the unitary group of degree n. We identify $\mathfrak{p}$ (resp. K) with T_n (resp. $U(n)$) through the map Ψ (resp. δ). We consider the action of $U(n)$ on T_n defined by

$$h\cdot\omega = h\omega\,{}^t h, \quad h\in U(n),\ \omega\in T_n. \tag{3.4}$$

Then the adjoint action (3.1) of K on $\mathfrak{p}$ is compatible with the action (3.4) of $U(n)$ on T_n through the map Ψ. Precisely for any $k\in K$ and $Z\in\mathfrak{p}$, we get

$$\Psi(k\,Z\,{}^t k) = \delta(k)\,\Psi(Z)\,{}^t\delta(k). \tag{3.5}$$

The action (3.4) induces the action of $U(n)$ on the polynomial algebra $\mathrm{Pol}(T_n)$ and the symmetric algebra $S(T_n)$ respectively. $\Longleftarrow$ +par

We denote by $\mathrm{Pol}(T_n)^{U(n)}$ $\Big($resp. $S(T_n)^{U(n)}\Big)$ the subalgebra of $\mathrm{Pol}(T_n)$ $\Big($resp. $S(T_n)\Big)$ consisting of $U(n)$-invariants. The following inner product $(\ ,\ )$ on T_n defined by

$$(Z,W) = \mathrm{tr}\big(Z\,\overline{W}\,\big), \quad Z,W\in T_n$$

gives an isomorphism as vector spaces

$$T_n\cong T_n^*, \quad Z\mapsto f_Z, \quad Z\in T_n, \tag{3.6}$$

where T_n^* denotes the dual space of T_n and f_Z is the linear functional on T_n defined by

$$f_Z(W) = (W,Z), \quad W\in T_n.$$

It is known that there is a canonical linear bijection of $S(T_n)^{U(n)}$ onto the algebra $\mathbb{D}(\mathbb{H}_n)$ of differential operators on $\mathbb{H}_n$ invariant under the action (1.1) of G. Identifying T_n with T_n^* by the above isomorphism (3.6), we get a canonical linear bijection

$$\Theta_n : \mathrm{Pol}(T_n)^{U(n)} \longrightarrow \mathbb{D}(\mathbb{H}_n) \tag{3.7}$$

of $\mathrm{Pol}(T_n)^{U(n)}$ onto $\mathbb{D}(\mathbb{H}_n)$. The map Θ_n is described explicitly as follows. Similarly the action (3.1) induces the action of K on the polynomial algebra $\mathrm{Pol}(\mathfrak{p})$ and the symmetric algebra $S(\mathfrak{p})$ respectively. Through the map Ψ, the subalgebra $\mathrm{Pol}(\mathfrak{p})^K$ of $\mathrm{Pol}(\mathfrak{p})$ consisting of K-invariants is isomorphic to $\mathrm{Pol}(T_n)^{U(n)}$. We put $N = n(n+1)$. Let $\{\xi_\alpha \mid 1\le\alpha\le N\}$ be a basis of a real vector space $\mathfrak{p}$. If $P\in\mathrm{Pol}(\mathfrak{p})^K$, then

$$\Big(\Theta_n(P)f\Big)(gK) = \left[P\left(\frac{\partial}{\partial t_\alpha}\right) f\left(g\exp\left(\sum_{\alpha=1}^N t_\alpha\xi_\alpha\right)K\right)\right]_{(t_\alpha)=0}, \tag{3.8}$$

where $f\in C^\infty(\mathbb{H}_n)$. We refer to [20, 21] for more detail. In general, it is hard to express $\Phi(P)$ explicitly for a polynomial $P\in\mathrm{Pol}(\mathfrak{p})^K$.

According to the work of Harish-Chandra [18, 19], the algebra $\mathbb{D}(\mathbb{H}_n)$ is generated by n algebraically independent generators and is isomorphic to the commutative algebra $\mathbb{C}[x_1,\dots,x_n]$ with n indeterminates. We note that n is the real rank of G. Let $\mathfrak{g}_{\mathbb{C}}$ be the complexification of $\mathfrak{g}$. It is known that $\mathbb{D}(\mathbb{H}_n)$ is isomorphic to the center of the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$.

Using a classical invariant theory (cf. [22, 61], we can show that $\mathrm{Pol}(T_n)^{U(n)}$ is generated by the following algebraically independent polynomials

$$q_j(\omega) = \mathrm{tr}\Big((\omega\overline{\omega})^j \Big), \quad \omega \in T_n, \quad j = 1, 2, \ldots, n. \tag{3.9}$$

For each j with $1 \leq j \leq n$, the image $\Theta_n(q_j)$ of q_j is an invariant differential operator on $\mathbb{H}_n$ of degree $2j$. The algebra $\mathbb{D}(\mathbb{H}_n)$ is generated by n algebraically independent generators $\Theta_n(q_1), \Theta_n(q_2), \ldots, \Theta_n(q_n)$. In particular,

$$\Theta_n(q_1) = c_1\, \mathrm{tr}\left(Y\, {}^t\!\left(Y \frac{\partial}{\partial \overline{\Omega}} \right) \frac{\partial}{\partial \Omega} \right) \quad \text{for some constant } c_1. \tag{3.10}$$

We observe that if we take $\omega = x + i\, y \in T_n$ with real x, y, then $q_1(\omega) = q_1(x, y) = \mathrm{tr}\big(x^2 + y^2\big)$ and

$$q_2(\omega) = q_2(x, y) = \mathrm{tr}\Big(\big(x^2 + y^2\big)^2 + 2\, x\big(xy - yx\big)y \Big).$$

It is a natural question to express the images $\Theta_n(q_j)$ explicitly for $j = 2, 3, \ldots, n$. We hope that the images $\Theta_n(q_j)$ for $j = 2, 3, \ldots, n$ are expressed in the form of the *trace* as $\Phi(q_1)$.

H. Maass [38] found algebraically independent generators $H_1, H_2, \ldots, H_n$ of $\mathbb{D}(\mathbb{H}_n)$. We will describe $H_1, H_2, \ldots, H_n$ explicitly. For $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$ and $\Omega = X + iY \in \mathbb{H}_n$ with real X, Y, we set

$$\Omega_* = M \cdot \Omega = X_* + iY_* \quad \text{with } X_*, Y_* \text{ real}.$$

We set

$$\begin{aligned} K &= \big(\Omega - \overline{\Omega}\, \big) \frac{\partial}{\partial \Omega} &&= 2\, i\, Y \frac{\partial}{\partial \Omega}, \\ \Lambda &= \big(\Omega - \overline{\Omega}\, \big) \frac{\partial}{\partial \overline{\Omega}} &&= 2\, i\, Y \frac{\partial}{\partial \overline{\Omega}}, \\ K_* &= \big(\Omega_* - \overline{\Omega}_*\, \big) \frac{\partial}{\partial \Omega_*} &&= 2\, i\, Y_* \frac{\partial}{\partial \Omega_*}, \\ \Lambda_* &= \big(\Omega_* - \overline{\Omega}_*\, \big) \frac{\partial}{\partial \overline{\Omega}_*} &&= 2\, i\, Y_* \frac{\partial}{\partial \overline{\Omega}_*}. \end{aligned}$$

Then it is easily seen that

$$K_* = {}^t(C\overline{\Omega} + D)^{-1}\, {}^t\big\{ (C\Omega + D)\, {}^tK \big\}, \tag{3.11}$$

$$\Lambda_* = {}^t(C\Omega + D)^{-1}\, {}^t\big\{ (C\overline{\Omega} + D)\, {}^t\Lambda \big\} \tag{3.12}$$

and

$${}^t\big\{ (C\overline{\Omega} + D)\, {}^t\Lambda \big\} = \Lambda\, {}^t(C\overline{\Omega} + D) - \frac{n+1}{2} \big(\Omega - \overline{\Omega}\, \big)\, {}^tC. \tag{3.13}$$

Using Formulas (3.11), (3.12) and (3.13), we can show that

$$\Lambda_* K_* + \frac{n+1}{2} K_* = {}^t(C\Omega + D)^{-1}\, {}^t\!\left\{ (C\Omega + D)\, {}^t\!\left(\Lambda K + \frac{n+1}{2} K \right) \right\}. \tag{3.14}$$

Therefore we get

$$\operatorname{tr}\Big(\Lambda_* K_* + \frac{n+1}{2} K_*\Big) = \operatorname{tr}\Big(\Lambda K + \frac{n+1}{2} K\Big). \tag{3.15}$$

We set

$$A^{(1)} = \Lambda K + \frac{n+1}{2} K. \tag{3.16}$$

We define $A^{(j)}$ $(j = 2, 3, \ldots, n)$ recursively by

$$\begin{aligned} A^{(j)} = {} & A^{(1)} A^{(j-1)} - \frac{n+1}{2} \Lambda A^{(j-1)} + \frac{1}{2} \Lambda \operatorname{tr}\Big(A^{(j-1)}\Big) \\ & + \frac{1}{2} \big(\Omega - \overline{\Omega}\big) \, {}^t\Big\{ \big(\Omega - \overline{\Omega}\big)^{-1} \, {}^t\Big({}^t\Lambda \, {}^t A^{(j-1)} \Big) \Big\}. \end{aligned} \tag{3.17}$$

We set

$$H_j = \operatorname{tr}\Big(A^{(j)}\Big), \quad j = 1, 2, \ldots, n. \tag{3.18}$$

As mentioned before, Maass proved that $H_1, H_2, \ldots, H_n$ are algebraically independent generators of $\mathbb{D}(\mathbb{H}_n)$.

In fact, we see that

$$-H_1 = \Delta_{n;1} = 4 \operatorname{tr}\Big(Y \, {}^t\Big(Y \frac{\partial}{\partial \overline{\Omega}} \Big) \frac{\partial}{\partial \Omega} \Big). \tag{3.19}$$

is the Laplacian for the invariant metric $ds^2_{n;1}$ on $\mathbb{H}_n$.

Example 3.1. We consider the case when $n = 1$. The algebra $\mathrm{Pol}(T_1)^{U(1)}$ is generated by the polynomial

$$q(\omega) = \omega \, \overline{\omega}, \quad \omega = x + iy \in \mathbb{C} \text{ with } x, y \text{ real}.$$

Using Formula (3.8), we get

$$\Theta_1(q) = 4 \, y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

Therefore $\mathbb{D}(\mathbb{H}_1) = \mathbb{C}\big[\Theta_1(q)\big] = \mathbb{C}[H_1]$.

Example 3.2. We consider the case when $n = 2$. The algebra $\mathrm{Pol}(T_2)^{U(2)}$ is generated by the polynomial

$$q_1(\omega) = \sigma\big(\omega \, \overline{\omega}\big), \quad q_2(\omega) = \sigma\Big(\big(\omega \, \overline{\omega}\big)^2\Big), \quad \omega \in T_2.$$

Using Formula (3.8), we may express $\Theta_2(q_1)$ and $\Theta_2(q_2)$ explicitly. $\Theta_2(q_1)$ is expressed by Formula (3.10). The computation of $\Theta_2(q_2)$ might be quite tedious. We leave the detail to the reader. In this case, $\Theta_2(q_2)$ was essentially computed in [11], Proposition 6. Therefore

$$\mathbb{D}(\mathbb{H}_2) = \mathbb{C}\big[\Theta_2(q_1), \Theta_2(q_2)\big] = \mathbb{C}[H_1, H_2].$$

In fact, the center of the universal enveloping algebra $\mathscr{U}(\mathfrak{g}_{\mathbb{C}})$ was computed in [11].

G. Shimura [56] found canonically defined algebraically independent generators of $\mathbb{D}(\mathbb{H}_n)$. We will describe his way of constructing those generators roughly.

Let $K_{\mathbb{C}}$, $\mathfrak{g}_{\mathbb{C}}$, $\mathfrak{k}_{\mathbb{C}}, \mathfrak{p}_{\mathbb{C}}, \dots$ denote the complexication of K, $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{p}, \dots$ respectively. Then we have the Cartan decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_{\mathbb{C}}, \quad \mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_{\mathbb{C}}^{+} + \mathfrak{p}_{\mathbb{C}}^{-}$$

with the properties

$$[\mathfrak{k}_{\mathbb{C}}, \mathfrak{p}_{\mathbb{C}}^{\pm}] \subset \mathfrak{p}_{\mathbb{C}}^{\pm}, \quad [\mathfrak{p}_{\mathbb{C}}^{+}, \mathfrak{p}_{\mathbb{C}}^{+}] = [\mathfrak{p}_{\mathbb{C}}^{-}, \mathfrak{p}_{\mathbb{C}}^{-}] = \{0\}, \quad [\mathfrak{p}_{\mathbb{C}}^{+}, \mathfrak{p}_{\mathbb{C}}^{-}] = \mathfrak{k}_{\mathbb{C}},$$

where

$$\mathfrak{g}_{\mathbb{C}} = \left\{ \begin{pmatrix} X_1 & X_2 \\ X_3 & -\,{}^tX_1 \end{pmatrix} \;\Big|\; X_1, X_2, X_3 \in \mathbb{C}^{(n,n)},\ X_2 = {}^tX_2,\ X_3 = {}^tX_3 \right\},$$

$$\mathfrak{k}_{\mathbb{C}} = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in \mathbb{C}^{(2n,2n)} \;\Big|\; {}^tA + A = 0,\ B = {}^tB \right\},$$

$$\mathfrak{p}_{\mathbb{C}} = \left\{ \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \in \mathbb{C}^{(2n,2n)} \;\Big|\; X = {}^tX,\ Y = {}^tY \right\},$$

$$\mathfrak{p}_{\mathbb{C}}^{+} = \left\{ \begin{pmatrix} Z & iZ \\ iZ & -Z \end{pmatrix} \in \mathbb{C}^{(2n,2n)} \;\Big|\; Z = {}^tZ \in \mathbb{C}^{(n,n)} \right\},$$

$$\mathfrak{p}_{\mathbb{C}}^{-} = \left\{ \begin{pmatrix} Z & -iZ \\ -iZ & -Z \end{pmatrix} \in \mathbb{C}^{(2n,2n)} \;\Big|\; Z = {}^tZ \in \mathbb{C}^{(n,n)} \right\}.$$

For a complex vector space W and a nonnegative integer r, we denote by $\mathrm{Pol}_r(W)$ the vector space of complex-valued homogeneous polynomial functions on W of degree r. We put

$$\mathrm{Pol}^r(W) := \sum_{s=0}^{r} \mathrm{Pol}_s(W).$$

$\mathrm{Ml}_r(W)$ denotes the vector space of all $\mathbb{C}$-multilinear maps of $W \times \cdots \times W$ (r copies) into $\mathbb{C}$. An element Q of $\mathrm{Ml}_r(W)$ is called *symmetric* if

$$Q(x_1, \dots, x_r) = Q(x_{\pi(1)}, \dots, x_{\pi(r)})$$

for each permutation π of $\{1, 2, \dots, r\}$. Given $P \in \mathrm{Pol}_r(W)$, there is a unique element symmetric element P_* of $\mathrm{Ml}_r(W)$ such that

$$P(x) = P_*(x, \dots, x) \qquad \text{for all } x \in W. \tag{3.20}$$

Moreover the map $P \mapsto P_*$ is a $\mathbb{C}$-linear bijection of $\mathrm{Pol}_r(W)$ onto the set of all symmetric elements of $\mathrm{Ml}_r(W)$. We let $S_r(W)$ denote the subspace consisting of all homogeneous elements of degree r in the symmetric algebra $S(W)$. We note that $\mathrm{Pol}_r(W)$ and $S_r(W)$ are dual to each other with respect to the pairing

$$\langle \alpha, x_1 \cdots x_r \rangle = \alpha_*(x_1, \dots, x_r) \qquad (x_i \in W,\ \alpha \in \mathrm{Pol}_r(W)). \tag{3.21}$$

Let $\mathfrak{p}_{\mathbb{C}}^*$ be the dual space of $\mathfrak{p}_{\mathbb{C}}$, that is, $\mathfrak{p}_{\mathbb{C}}^* = \mathrm{Pol}_1(\mathfrak{p}_{\mathbb{C}})$. Let $\{X_1, \dots, X_N\}$ be a basis of $\mathfrak{p}_{\mathbb{C}}$ and $\{Y_1, \dots, Y_N\}$ be the basis of $\mathfrak{p}_{\mathbb{C}}^*$ dual to $\{X_\nu\}$, where $N = n(n+1)$. We note that $\mathrm{Pol}_r(\mathfrak{p}_{\mathbb{C}})$ and $\mathrm{Pol}_r(\mathfrak{p}_{\mathbb{C}}^*)$ are dual to each other with respect to the pairing

$$\langle \alpha, \beta \rangle = \sum \alpha_*(X_{i_1}, \dots, X_{i_r})\, \beta_*(Y_{i_1}, \dots, Y_{i_r}), \tag{3.22}$$

where $\alpha \in \mathrm{Pol}_r(\mathfrak{p}_{\mathbb{C}})$, $\beta \in \mathrm{Pol}_r(\mathfrak{p}_{\mathbb{C}}^*)$ and $(i_1, \ldots, i_r)$ runs over $\{1, \ldots, N\}^r$. Let $\mathscr{U}(\mathfrak{g}_{\mathbb{C}})$ be the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$ and $\mathscr{U}^p(\mathfrak{g}_{\mathbb{C}})$ its subspace spanned by the elements of the form $V_1 \cdots V_s$ with $V_i \in \mathfrak{g}_{\mathbb{C}}$ and $s \le p$. We recall that there is a $\mathbb{C}$-linear bijection ψ of the symmetric algebra $S(\mathfrak{g}_{\mathbb{C}})$ of $\mathfrak{g}_{\mathbb{C}}$ onto $\mathscr{U}(\mathfrak{g}_{\mathbb{C}})$ which is characterized by the property that $\psi(X^r) = X^r$ for all $X \in \mathfrak{g}_{\mathbb{C}}$. For each $\alpha \in \mathrm{Pol}_r(\mathfrak{p}_{\mathbb{C}}^*)$ we define an element $\omega(\alpha)$ of $\mathscr{U}(\mathfrak{g}_{\mathbb{C}})$ by

$$\omega(\alpha) := \sum \alpha_*(Y_{i_1}, \ldots, Y_{i_r})\, X_{i_1} \cdots X_{i_r}, \tag{3.23}$$

where $(i_1, \ldots, i_r)$ runs over $\{1, \ldots, N\}^r$. If $Y \in \mathfrak{p}_{\mathbb{C}}$, then Y^r as an element of $\mathrm{Pol}_r(\mathfrak{p}_{\mathbb{C}}^*)$ is defined by

$$Y^r(u) = Y(u)^r \qquad \text{for all } u \in \mathfrak{p}_{\mathbb{C}}^*.$$

Hence $(Y^r)_*(u_1, \ldots, u_r) = Y(u_1) \cdots Y(u_r)$. According to (2.25), we see that if $\alpha(\sum t_i Y_i) = P(t_1, \ldots, t_N)$ for $t_i \in \mathbb{C}$ with a polynomial P, then

$$\omega(\alpha) = \psi(P(X_1, \ldots, X_N)). \tag{3.24}$$

Thus ω is a $\mathbb{C}$-linear injection of $\mathrm{Pol}(\mathfrak{p}_{\mathbb{C}}^*)$ into $\mathscr{U}(\mathfrak{g}_{\mathbb{C}})$ independent of the choice of a basis. We observe that $\omega\big(\mathrm{Pol}_r(\mathfrak{p}_{\mathbb{C}}^*)\big) = \psi(S_r(\mathfrak{p}_{\mathbb{C}}))$. It is a well-known fact that if $\alpha_1, \ldots, \alpha_m \in \mathrm{Pol}_r(\mathfrak{p}_{\mathbb{C}}^*)$, then

$$\omega(\alpha_1 \cdots \alpha_m) - \omega(\alpha_m) \cdots \omega(\alpha_1) \in \mathscr{U}^{r-1}(\mathfrak{g}_{\mathbb{C}}). \tag{3.25}$$

We have a canonical pairing

$$\langle\ ,\ \rangle : \mathrm{Pol}_r(\mathfrak{p}_{\mathbb{C}}^+) \times \mathrm{Pol}_r(\mathfrak{p}_{\mathbb{C}}^-) \longrightarrow \mathbb{C} \tag{3.26}$$

defined by

$$\langle f, g\rangle = \sum f_*(\widetilde{X}_{i_1}, \ldots, \widetilde{X}_{i_r}) g_*(\widetilde{Y}_{i_1}, \ldots, \widetilde{Y}_{i_r}), \tag{3.27}$$

where f_* (resp. g_*) are the unique symmetric elements of $\mathrm{Ml}_r(\mathfrak{p}_{\mathbb{C}}^+)$ (resp. $\mathrm{Ml}_r(\mathfrak{p}_{\mathbb{C}}^-)$), and $\{\widetilde{X}_1, \ldots, \widetilde{X}_{\widetilde{N}}\}$ and $\{\widetilde{Y}_1, \ldots, \widetilde{Y}_{\widetilde{N}}\}$ are dual bases of $\mathfrak{p}_{\mathbb{C}}^+$ and $\mathfrak{p}_{\mathbb{C}}^-$ with respect to the Killing form $B(X, Y) = 2(n+1)\,\mathrm{tr}(XY)$, $\widetilde{N} = \frac{n(n+1)}{2}$, and $(i_1, \ldots, i_r)$ runs over $\big\{1, \ldots, \widetilde{N}\big\}^r$.

The adjoint representation of $K_{\mathbb{C}}$ on $\mathfrak{p}_{\mathbb{C}}^{\pm}$ induces the representation of $K_{\mathbb{C}}$ on $\mathrm{Pol}_r(\mathfrak{p}_{\mathbb{C}}^{\pm})$. Given a $K_{\mathbb{C}}$-irreducible subspace Z of $\mathrm{Pol}_r(\mathfrak{p}_{\mathbb{C}}^+)$, we can find a unique $K_{\mathbb{C}}$-irreducible subspace W of $\mathrm{Pol}_r(\mathfrak{p}_{\mathbb{C}}^-)$ such that $\mathrm{Pol}_r(\mathfrak{p}_{\mathbb{C}}^-)$ is the direct sum of W and the annihilator of Z. Then Z and W are dual with respect to the pairing (3.26). Take bases $\{\zeta_1, \ldots, \zeta_\kappa\}$ of Z and $\{\xi_1, \ldots, \xi_\kappa\}$ of W that are dual to each other. We set

$$f_Z(x, y) = \sum_{\nu=1}^{\kappa} \zeta_\nu(x)\, \xi_\nu(y) \qquad (x \in \mathfrak{p}_{\mathbb{C}}^+,\ y \in \mathfrak{p}_{\mathbb{C}}^-). \tag{3.28}$$

It is easily seen that f_Z belongs to $\mathrm{Pol}_{2r}(\mathfrak{p}_{\mathbb{C}})^K$ and is independent of the choice of dual bases $\{\zeta_\nu\}$ and $\{\xi_\nu\}$. Shimura [56] proved that there exists a canonically defined set $\{Z_1, \ldots, Z_n\}$ with a $K_{\mathbb{C}}$-irreducible subspace Z_r of $\mathrm{Pol}_r(\mathfrak{p}_{\mathbb{C}}^+)$ $(1 \le r \le n)$ such that $f_{Z_1}, \ldots, f_{Z_n}$ are algebraically independent generators of $\mathrm{Pol}(\mathfrak{p}_{\mathbb{C}})^K$. We can identify $\mathfrak{p}_{\mathbb{C}}^+$ with T_n. We recall that T_n denotes the vector space of $n \times n$ symmetric complex matrices. We can take Z_r as the subspace of $\mathrm{Pol}_r(T_n)$ spanned by the functions $f_{a;r}(Z) = \det_r({}^t a Z a)$ for all $a \in GL(n, \mathbb{C})$, where $\det_r(x)$ denotes the determinant of the upper left $r \times r$ submatrix of x. For every $f \in \mathrm{Pol}(\mathfrak{p}_{\mathbb{C}})^K$, we let $\Omega(f)$ denote the element of $\mathbb{D}(\mathbb{H}_n)$ represented by $\omega(f)$. Then $\mathbb{D}(\mathbb{H}_n)$ is the polynomial ring $\mathbb{C}[\omega(f_{Z_1}), \ldots, \omega(f_{Z_n})]$ generated by n algebraically independent elements $\omega(f_{Z_1}), \ldots, \omega(f_{Z_n})$.

Now we investigate differential operators on the Siegel–Jacobi space $\mathbb{H}_{n,m}$ invariant under the action (1.2) of G^J. The stabilizer K^J of G^J at $(iI_n, 0)$ is given by

$$K^J = \left\{ \big(k, (0, 0; \kappa)\big) \;\middle|\; k \in K,\ \kappa = {}^t\kappa \in \mathbb{R}^{(m,m)} \right\}.$$

Therefore $\mathbb{H}_n \times \mathbb{C}^{(m,n)} \cong G^J/K^J$ is a homogeneous space which is not symmetric. The Lie algebra $\mathfrak{g}^J$ of G^J has a decomposition

$$\mathfrak{g}^J = \mathfrak{k}^J + \mathfrak{p}^J,$$

where

$$\begin{aligned}
\mathfrak{g}^J &= \left\{ \big(Z, (P, Q, R)\big) \;\middle|\; Z \in \mathfrak{g},\ P, Q \in \mathbb{R}^{(m,n)},\ R = {}^tR \in \mathbb{R}^{(m,m)} \right\}, \\
\mathfrak{k}^J &= \left\{ \big(X, (0, 0, R)\big) \;\middle|\; X \in \mathfrak{k},\ R = {}^tR \in \mathbb{R}^{(m,m)} \right\}, \\
\mathfrak{p}^J &= \left\{ \big(Y, (P, Q, 0)\big) \;\middle|\; Y \in \mathfrak{p},\ P, Q \in \mathbb{R}^{(m,n)} \right\}.
\end{aligned}$$

Thus the tangent space of the homogeneous space $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$ at $(iI_n, 0)$ is identified with $\mathfrak{p}^J$.

If $\alpha = \left(\begin{pmatrix} X_1 & Y_1 \\ Z_1 & -{}^tX_1 \end{pmatrix}, (P_1, Q_1, R_1) \right)$ and $\beta = \left(\begin{pmatrix} X_2 & Y_2 \\ Z_2 & -{}^tX_2 \end{pmatrix}, (P_2, Q_2, R_2) \right)$ are elements of $\mathfrak{g}^J$, then the Lie bracket $[\alpha, \beta]$ of α and β is given by

$$[\alpha, \beta] = \left(\begin{pmatrix} X^* & Y^* \\ Z^* & -{}^tX^* \end{pmatrix}, (P^*, Q^*, R^*) \right), \tag{3.29}$$

where

$$\begin{aligned}
X^* &= X_1X_2 - X_2X_1 + Y_1Z_2 - Y_2Z_1, \\
Y^* &= X_1Y_2 - X_2Y_1 + Y_2\,{}^tX_1 - Y_1\,{}^tX_2, \\
Z^* &= Z_1X_2 - Z_2X_1 + {}^tX_2Z_1 - {}^tX_1Z_2, \\
P^* &= P_1X_2 - P_2X_1 + Q_1Z_2 - Q_2Z_1, \\
Q^* &= P_1Y_2 - P_2Y_1 + Q_2\,{}^tX_1 - Q_1\,{}^tX_2, \\
R^* &= P_1\,{}^tQ_2 - P_2\,{}^tQ_1 + Q_2\,{}^tP_1 - Q_1\,{}^tP_2
\end{aligned}$$

Lemma 3.1.

$$[\mathfrak{k}^J, \mathfrak{k}^J] \subset \mathfrak{k}^J, \quad [\mathfrak{k}^J, \mathfrak{p}^J] \subset \mathfrak{p}^J.$$

Proof. The proof follows immediately from Formula (3.29). □

Lemma 3.2. *Let*

$$k^J = \left(\begin{pmatrix} A & -B \\ B & A \end{pmatrix}, (0,0,\kappa) \right) \in K^J$$

with $\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in K,\ \kappa = {}^t\kappa \in \mathbb{R}^{(m,m)}$ *and*

$$\alpha = \left(\begin{pmatrix} X & Y \\ Y & -X \end{pmatrix}, (P,Q,0) \right) \in \mathfrak{p}^J$$

with $X = {}^tX,\ Y = {}^tY \in \mathbb{R}^{(n,n)},\ P,Q \in \mathbb{R}^{(m,n)}$. *Then the adjoint action of* K^J *on* $\mathfrak{p}^J$ *is given by*

$$Ad(k^J)\alpha = \left(\begin{pmatrix} X_* & Y_* \\ Y_* & -X_* \end{pmatrix}, (P_*,Q_*,0) \right), \tag{3.30}$$

where

$$X_* = AX\,{}^tA - \big(BX\,{}^tB + BY\,{}^tA + AY\,{}^tB\big), \tag{3.31}$$

$$Y_* = \big(AX\,{}^tB + AY\,{}^tA + BX\,{}^tA\big) - BY\,{}^tB, \tag{3.32}$$

$$P_* = P\,{}^tA - Q\,{}^tB, \tag{3.33}$$

$$Q_* = P\,{}^tB + Q\,{}^tA. \tag{3.34}$$

Proof. We leave the proof to the reader. □

We recall that T_n denotes the vector space of all $n \times n$ symmetric complex matrices. For brevity, we put $T_{n,m} := T_n \times \mathbb{C}^{(m,n)}$. We define the real linear isomorphism $\Phi : \mathfrak{p}^J \longrightarrow T_{n,m}$ by

$$\Phi\left(\left(\begin{pmatrix} X & Y \\ Y & -X \end{pmatrix}, (P,Q,0) \right) \right) = \big(X + i\,Y,\ P + i\,Q\big), \tag{3.35}$$

where $\begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \in \mathfrak{p}$ and $P,Q \in \mathbb{R}^{(m,n)}$.

Let $S(m,\mathbb{R})$ denote the additive group consisting of all $m \times m$ real symmetric matrices. Now we define the isomorphism $\theta : K^J \longrightarrow U(n) \times S(m,\mathbb{R})$ by

$$\theta(h,(0,0,\kappa)) = (\delta(h),\kappa), \quad h \in K,\ \kappa \in S(m,\mathbb{R}), \tag{3.36}$$

where $\delta : K \longrightarrow U(n)$ is the map defined by (3.3). Identifying $\mathbb{R}^{(m,n)} \times \mathbb{R}^{(m,n)}$ with $\mathbb{C}^{(m,n)}$, we can identify $\mathfrak{p}^J$ with $T_n \times \mathbb{C}^{(m,n)}$.

Theorem 3.1. *The adjoint representation of* K^J *on* $\mathfrak{p}^J$ *is compatible with the natural action of* $U(n) \times S(m,\mathbb{R})$ *on* $T_{n,m}$ *defined by*

$$(h,\kappa)\cdot(\omega,z) := (h\,\omega\,{}^th,\ z\,{}^th), \qquad h \in U(n),\ \kappa \in S(m,\mathbb{R}),\ (\omega,z) \in T_{n,m} \tag{3.37}$$

through the maps Φ *and* θ. *Precisely, if* $k^J \in K^J$ *and* $\alpha \in \mathfrak{p}^J$, *then we have the following equality*

$$\Phi\big(Ad\big(k^J\big)\alpha\big) = \theta\big(k^J\big)\cdot\Phi(\alpha). \tag{3.38}$$

Here we regard the complex vector space $T_{n,m}$ *as a real vector space.*

Proof. Let

$$k^J = \left(\begin{pmatrix} A & -B \\ B & A \end{pmatrix}, (0,0,\kappa) \right) \in K^J$$

with $\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in K,\ \kappa = {}^t\kappa \in \mathbb{R}^{(m,m)}$ and

$$\alpha = \left(\begin{pmatrix} X & Y \\ Y & -X \end{pmatrix}, (P,Q,0) \right) \in \mathfrak{p}^J$$

with $X = {}^tX,\ Y = {}^tY \in \mathbb{R}^{(n,n)},\ P, Q \in \mathbb{R}^{(m,n)}$. Then we have

$$\begin{aligned} \theta\big(k^J\big)\cdot\Phi(\alpha) &= \big(A + i\,B,\,\kappa\big)\cdot\big(X + i\,Y,\, P + i\,Q\big) \\ &= \big((A+iB)(X+iY)\,{}^t(A+iB),\,(P+iQ)\,{}^t(A+iB)\big) \\ &= \big(X_* + i\,Y_*,\, P_* + i\,Q_*\big) \\ &= \Phi\left(\begin{pmatrix} X_* & Y_* \\ Y_* & -X_* \end{pmatrix}, (P_*, Q_*, 0) \right) \\ &= \Phi\big(Ad\big(k^J\big)\alpha\big) \qquad \text{(by Lemma 3.2)}, \end{aligned}$$

where X_*, Y_*, Z_* and Q_* are given by the formulas (3.31), (3.32), (3.33) and (3.34) respectively. □

We now study the algebra $\mathbb{D}(\mathbb{H}_{n,m})$ of all differential operators on $\mathbb{H}_{n,m}$ invariant under the *natural action* (1.2) of G^J. The action (3.37) induces the action of $U(n)$ on the polynomial algebra $\mathrm{Pol}_{n,m} := \mathrm{Pol}\,(T_{n,m})$. We denote by $\mathrm{Pol}_{n,m}^{U(n)}$ the subalgebra of $\mathrm{Pol}_{n,m}$ consisting of all $U(n)$-invariants. Similarly the action (3.30) of K induces the action of K on the polynomial algebra $\mathrm{Pol}\big(\mathfrak{p}^J\big)$. We see that through the identification of $\mathfrak{p}^J$ with $T_{n,m}$, the algebra $\mathrm{Pol}\big(\mathfrak{p}^J\big)$ is isomorphic to $\mathrm{Pol}_{n,m}$. The following $U(n)$-invariant inner product $(\ ,\ )_*$ of the complex vector space $T_{n,m}$ defined by

$$\big((\omega,z),(\omega',z')\big)_* = \mathrm{tr}\big(\omega\overline{\omega'}\big) + \mathrm{tr}\big(z\,{}^t\overline{z'}\big), \quad (\omega,z),\,(\omega',z') \in T_{n,m}$$

gives a canonical isomorphism

$$T_{n,m} \cong T_{n,m}^*, \quad (\omega,z) \mapsto f_{\omega,z}, \quad (\omega,z) \in T_{n,m},$$

where $f_{\omega,z}$ is the linear functional on $T_{n,m}$ defined by

$$f_{\omega,z}\big((\omega',z')\big) = \big((\omega',z'),(\omega,z)\big)_*, \quad (\omega',z') \in T_{n,m}.$$

According to Helgason ([21], p. 287), one gets a canonical linear bijection of $S(T_{n,m})^{U(n)}$ onto $\mathbb{D}(\mathbb{H}_{n,m})$. Identifying $T_{n,m}$ with $T_{n,m}^*$ by the above isomorphism, one gets a natural linear bijection

$$\Theta_{n,m} :\ \mathrm{Pol}_{n,m}^{U(n)} \longrightarrow \mathbb{D}(\mathbb{H}_{n,m})$$

of $\mathrm{Pol}_{n,m}^{U(n)}$ onto $\mathbb{D}(\mathbb{H}_{n,m})$. The map $\Theta_{n,m}$ is described explicitly as follows. We put $N_\star = n(n+1)+2mn$. Let $\{\eta_\alpha \mid 1 \le \alpha \le N_\star\}$ be a basis of $\mathfrak{p}^J$. If $P \in \mathrm{Pol}(\mathfrak{p}^J)^K = \mathrm{Pol}_{n,m}^{U(n)}$, then

$$\Big(\Theta_{n,m}(P)f\Big)(gK^J) = \left[P\left(\frac{\partial}{\partial t_\alpha}\right) f\left(g\exp\left(\sum_{\alpha=1}^{N_\star} t_\alpha \eta_\alpha\right)K^J\right)\right]_{(t_\alpha)=0}, \tag{3.39}$$

where $g \in G^J$ and $f \in C^\infty(\mathbb{H}_{n,m})$. In general, it is hard to express $\Theta_{n,m}(P)$ explicitly for a polynomial $P \in \mathrm{Pol}(\mathfrak{p}^J)^K$.

We propose the following natural problems.

Problem 1. Find a complete list of explicit generators of $\mathrm{Pol}_{n,m}^{U(n)}$.

Problem 2. Find all the relations among a set of generators of $\mathrm{Pol}_{n,m}^{U(n)}$.

Problem 3. Find an easy or effective way to express the images of the above invariant polynomials or generators of $\mathrm{Pol}_{n,m}^{U(n)}$ under the Helgason map $\Theta_{n,m}$ explicitly.

Problem 4. Find a complete list of explicit generators of the algebra $\mathbb{D}(\mathbb{H}_{n,m})$. Or construct explicit G^J-invariant differential operators on $\mathbb{H}_{n,m}$.

Problem 5. Find all the relations among a set of generators of $\mathbb{D}(\mathbb{H}_{n,m})$.

Problem 6. Is $\mathrm{Pol}_{n,m}^{U(n)}$ finitely generated?

Problem 7. Is $\mathbb{D}(\mathbb{H}_{n,m})$ finitely generated?

We will give answers to Problems 1, 2 and 6.

We put $\varphi^{(2k)} = \mathrm{tr}((w\bar{w})^k)$. Moreover, for $1 \le a, b \le m$ and $k \ge 0$, we put

$$\psi_{ba}^{(0,2k,0)} = (\bar{z}(w\bar{w})^k\, {}^t z)_{ba}, \qquad \psi_{ba}^{(1,2k,0)} = (z\bar{w}(w\bar{w})^k\, {}^t z)_{ba},$$
$$\psi_{ba}^{(0,2k,1)} = (\bar{z}(w\bar{w})^k w\, {}^t\bar{z})_{ba}, \qquad \psi_{ba}^{(1,2k,1)} = (z\bar{w}(w\bar{w})^k w\, {}^t\bar{z})_{ba}.$$

Then we have the following relations:

$$\varphi^{(2k)} = \bar{\varphi}^{(2k)},\ \psi_{ab}^{(1,2k,1)} = \psi_{ba}^{(0,2k+2,0)},\ \psi_{ab}^{(1,2k,0)} = \psi_{ba}^{(1,2k,0)} = \bar{\psi}_{ab}^{(0,2k,1)} = \bar{\psi}_{ba}^{(0,2k,1)}. \tag{3.40}$$

Then we have the following theorem:

Theorem 3.2. *The algebra* $\mathrm{Pol}_{n,m}^{U(n)}$ *is generated by the following polynomials*:

$$\varphi^{(2k+2)}, \qquad \mathrm{Re}\,\psi_{ab}^{(0,2k,0)}, \qquad \mathrm{Im}\,\psi_{cd}^{(0,2k,0)}, \qquad \mathrm{Re}\,\psi_{ab}^{(1,2k,0)}, \qquad \mathrm{Im}\,\psi_{ab}^{(1,2k,0)}.$$

Here the indices run as follows:

$$0 \le k \le n-1, \qquad 1 \le a \le b \le m, \qquad 1 \le c < d \le m.$$

This is seen from the following theorem by using (3.40):

Theorem 3.3. *The algebra* $\mathrm{Pol}_{n,m}^{U(n)}$ *is generated by* $\varphi^{(2k+2)}$, $\psi_{ba}^{(0,2k,0)}$, $\psi_{ba}^{(0,2k,1)}$, *and* $\psi_{ba}^{(1,2k,0)}$. *Here the indices run as follows*:

$$0 \le k \le n-1, \qquad 1 \le a, b \le m.$$

Proof. See Theorem 3.3 in [26]. □

Problem 2, that is, the second fundamental theorem for $\mathrm{Pol}_{n,m}^{U(n)}$ is stated as follows. We consider indeterminates $\tilde{\omega}^{(2k+2)}$ and $\tilde{\psi}_{ba}^{(\varepsilon,2k,\varepsilon')}$ corresponding to $\omega^{(2k+2)}$ and $\psi_{ba}^{(\varepsilon,2k,\varepsilon')}$, respectively. For these, we assume the relations

$$\tilde{\psi}_{ba}^{(1,2k,1)} = \tilde{\psi}_{ab}^{(0,2k+2,0)}, \qquad \tilde{\psi}_{ab}^{(1,2k,0)} = \tilde{\psi}_{ba}^{(1,2k,0)}, \qquad \tilde{\psi}_{ab}^{(0,2k,1)} = \tilde{\psi}_{ba}^{(0,2k,1)}.$$

We denote by $\tilde{\mathcal{Q}}$ the polynomial algebra in the following indeterminates:

$$\tilde{\omega}^{(2k+2)}, \quad \tilde{\psi}_{ba}^{(0,2k,0)}, \quad \tilde{\psi}_{ba}^{(0,2k,1)}, \quad \tilde{\psi}_{ba}^{(1,2k,0)}.$$

Here the indices run as follows:

$$0 \le k \le n-1, \qquad 1 \le a, b \le m.$$

The relations among the generators of $\mathrm{Pol}_{n,m}^{U(n)}$ are described as follows:

Theorem 3.4. *The kernel of the natural map from* $\tilde{\mathcal{Q}}$ *to* $\mathrm{Pol}_{n,m}^{U(n)}$ *is generated by the entries of* $A_{(c,\varepsilon),(c',\varepsilon')}^{(q)} B^{(q)}$ *with*

$$q \in \{2, 3, \ldots, n+1\}, \quad \varepsilon = (\varepsilon_1, \ldots, \varepsilon_q), \quad \varepsilon' = (\varepsilon_1', \ldots, \varepsilon_q') \in \{0,1\}^q,$$
$$c = (c_1, \ldots, c_q), \quad c' = (c_1', \ldots, c_q') \in \{1, \ldots, m\}^q.$$

Here the notation is as follows. We put

$$\Delta_{(c,\varepsilon),(c',\varepsilon')}^{(q),(\lambda_1,\lambda_2)} = \sum_{l_1+\cdots+l_q=\lambda_1+\lambda_2} K_{(\lambda_1,\lambda_2),(l_1,\ldots,l_q)} \sum_{\sigma\in S_q} \mathrm{sgn}(\sigma) \tilde{\psi}_{c_1 c'_{\sigma(1)}}^{(\varepsilon_1,2l_1,\varepsilon_1')} \cdots \tilde{\psi}_{c_q c'_{\sigma(q)}}^{(\varepsilon_q,2l_q,\varepsilon_q')}.$$

Here $K_{\lambda,\mu}$ means the Kostka number. Namely, in general, we define $K_{\lambda,\mu}$ by

$$s_\lambda(u_1, \ldots, u_q) = \sum_{l_1,\ldots,l_q\ge 0} K_{\lambda,(l_1,\ldots,l_q)} u_1^{l_1} \cdots u_q^{l_q},$$

where s_λ is the Schur polynomial. In other words, $\Delta_{(c,\varepsilon),(c',\varepsilon')}^{(q),(\lambda_1,\lambda_2)}$ is the image of the Schur polynomial $s_{(\lambda_1,\lambda_2)}(u_1, \ldots, u_q)$ under the linear map

$$u_1^{l_1} \cdots u_q^{l_q} \mapsto \sum_{\sigma\in S_q} \mathrm{sgn}(\sigma) \tilde{\psi}_{c_1 c'_{\sigma(1)}}^{(\varepsilon_1,2l_1,\varepsilon_1')} \cdots \tilde{\psi}_{c_q c'_{\sigma(q)}}^{(\varepsilon_q,2l_q,\varepsilon_q')}.$$

Moreover we replace $\tilde{\psi}_{ba}^{(1,2n-2,1)}$ in $\Delta_{(c,\varepsilon),(c',\varepsilon')}^{(q),(\lambda_1,\lambda_2)}$ by

$$\sum_{k=1}^{n} (-1)^{k-1} \tilde{\omega}^{(2k)} \tilde{\psi}_{ab}^{0,2n-2k,0}.$$

Finally $A^{(q)}_{(c,\varepsilon),(c',\varepsilon')}$ and $B^{(q)}$ are the following matrices (an alternating matrix of size $q'+2$ and a $(q'+2)\times 1$ matrix):

$$A^{(q)}_{(c,\varepsilon),(c',\varepsilon')} = \begin{pmatrix} 0 & \Delta^{(q),(0,0)}_{(c,\varepsilon),(c',\varepsilon')} & \Delta^{(q),(1,0)}_{(c,\varepsilon),(c',\varepsilon')} & \cdots & \Delta^{(q),(q',0)}_{(c,\varepsilon),(c',\varepsilon')} \\ -\Delta^{(q),(0,0)}_{(c,\varepsilon),(c',\varepsilon')} & 0 & \Delta^{(q),(1,1)}_{(c,\varepsilon),(c',\varepsilon')} & \cdots & \Delta^{(q),(q',1)}_{(c,\varepsilon),(c',\varepsilon')} \\ -\Delta^{(q),(1,0)}_{(c,\varepsilon),(c',\varepsilon')} & -\Delta^{(r),(1,1)}_{(c,\varepsilon),(c',\varepsilon')} & 0 & \cdots & \Delta^{(q),(q',2)}_{(c,\varepsilon),(c',\varepsilon')} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\Delta^{(q),(q',0)}_{(c,\varepsilon),(c',\varepsilon')} & -\Delta^{(q),(q',1)}_{(c,\varepsilon),(c',\varepsilon')} & -\Delta^{(q),(c',2)}_{(c,\varepsilon),(c',\varepsilon')} & \cdots & 0 \end{pmatrix},$$

$$B^{(q)} = \begin{pmatrix} (-1)^{q'}\tilde{\omega}^{(q')} \\ \vdots \\ \tilde{\omega}^{(2)} \\ -\tilde{\omega}^{(1)} \\ \tilde{\omega}^{(0)} \end{pmatrix}.$$

Here we put $q' = n+1-q$.

The proof of Theorem 3.4 is complicated, but it is deduced from the second fundamental theorem of invariant theory for vector invariants (this is quite parallel with the fact that Theorem 3.3 follows from the first fundamental theorem of invariant theory for vector invariants). The detail will be given in the forthcoming paper.

Remark 3.1. Itoh, Ochiai and Yang [26] solved all the problems (Problem 1–Problem 7) proposed in this section when $n = m = 1$.

We present some interesting $U(n)$-invariants. For an $m\times m$ matrix S, we define the following invariant polynomials in $\mathrm{Pol}_{n,m}^{U(n)}$:

$$\begin{aligned} m^{(1)}_{j;S}(\omega,z) &= \mathrm{Re}\left(\mathrm{tr}\big(\omega\overline{\omega} + {}^t z S\overline{z}\big)^j\right), \quad 1\le j\le n, \\ m^{(2)}_{j;S}(\omega,z) &= \mathrm{Im}\left(\mathrm{tr}\big(\omega\overline{\omega} + {}^t z S\overline{z}\big)^j\right), \quad 1\le j\le n, \\ q^{(1)}_{k;S}(\omega,z) &= \mathrm{Re}\left(\mathrm{tr}\big(({}^t z\, S\,\overline{z})^k\big)\right), \qquad 1\le k\le m, \\ q^{(2)}_{k;S}(\omega,z) &= \mathrm{Im}\left(\mathrm{tr}\big(({}^t z\, S\,\overline{z})^k\big)\right), \qquad 1\le k\le m, \\ \theta^{(1)}_{i,k,j;S}(\omega,z) &= \mathrm{Re}\left(\mathrm{tr}\big((\omega\overline{\omega})^i\,({}^t z\, S\,\overline{z})^k\,(\omega\overline{\omega} + {}^t z\, S\,\overline{z})^j\,\big)\right), \\ \theta^{(2)}_{i,k,j;S}(\omega,z) &= \mathrm{Im}\left(\mathrm{tr}\big((\omega\overline{\omega})^i\,({}^t z\, S\,\overline{z})^k\,(\omega\overline{\omega} + {}^t z\, S\,\overline{z})^j\,\big)\right), \end{aligned}$$

where $1\le i,j\le n$ and $1\le k\le m$.

We define the following $U(n)$-invariant polynomials in $\mathrm{Pol}_{n,m}^{U(n)}$.

$$r_{jk}^{(1)}(\omega,z) = \mathrm{Re}\Big(\det\big((\omega\overline{\omega})^j\,({}^tz\overline{z})^k\,\big)\Big),\quad 1\le j\le n,\ 1\le k\le m,$$
$$r_{jk}^{(2)}(\omega,z) = \mathrm{Im}\Big(\det\big((\omega\overline{\omega})^j\,({}^tz\overline{z})^k\,\big)\Big),\quad 1\le j\le n,\ 1\le k\le m.$$

4. The partial Cayley transform

Let

$$\mathbb{D}_n = \Big\{ W\in\mathbb{C}^{(n,n)}\mid W={}^tW,\ I_n-\overline{W}W>0\Big\}$$

be the generalized unit disk. We set

$$G_* = T^{-1}Sp(n,\mathbb{R})\,T,\qquad T:=\frac{1}{\sqrt{2}}\begin{pmatrix} I_n & I_n\\ iI_n & -iI_n\end{pmatrix}.$$

It is easily seen that

$$G_* = \left\{ \begin{pmatrix} P & Q\\ \overline{Q} & \overline{P}\end{pmatrix}\in\mathbb{C}^{(2n,2n)}\ \middle|\ {}^tP\overline{P}-{}^t\overline{Q}Q=I_n,\ {}^tP\overline{Q}={}^t\overline{Q}P\right\}.$$

Then G_* acts on $\mathbb{D}_n$ transitively by

$$\begin{pmatrix} P & Q\\ \overline{Q} & \overline{P}\end{pmatrix}\cdot W=(PW+Q)(\overline{Q}W+\overline{P})^{-1},\qquad \begin{pmatrix} P & Q\\ \overline{Q} & \overline{P}\end{pmatrix}\in G_*,\ W\in\mathbb{D}_n. \tag{4.1}$$

It is well known that the action (1.1) is compatible with the action (4.1) through the Cayley transform $\Phi:\mathbb{D}_n\longrightarrow\mathbb{H}_n$ given by

$$\Phi(W):=i\,(I_n+W)(I_n-W)^{-1},\qquad W\in\mathbb{D}_n. \tag{4.2}$$

In other words, if $M\in Sp(n,\mathbb{R})$ and $W\in\mathbb{D}_n$, then

$$M\cdot\Phi(W)=\Phi(M_*\cdot W), \tag{4.3}$$

where $M_*=T^{-1}MT$. We refer to [31] for generalized Cayley transforms of bounded symmetric domains.

For brevity, we write $\mathbb{D}_{n,m}:=\mathbb{D}_n\times\mathbb{C}^{(m,n)}$. This homogeneous space $\mathbb{D}_{n,m}$ is called the *Siegel–Jacobi disk* of degree n and index m. For a coordinate $(W,\eta)\in\mathbb{D}_{n,m}$ with $W=(w_{\mu\nu})\in\mathbb{D}_n$ and $\eta=(\eta_{kl})\in\mathbb{C}^{(m,n)}$, we put

$$dW=(dw_{\mu\nu}),\qquad d\overline{W}=(d\overline{w}_{\mu\nu}),$$
$$d\eta=(d\eta_{kl}),\qquad d\overline{\eta}=(d\overline{\eta}_{kl})$$

and

$$\frac{\partial}{\partial W}=\left(\frac{1+\delta_{\mu\nu}}{2}\,\frac{\partial}{\partial w_{\mu\nu}}\right),\qquad \frac{\partial}{\partial\overline{W}}=\left(\frac{1+\delta_{\mu\nu}}{2}\,\frac{\partial}{\partial\overline{w}_{\mu\nu}}\right),$$

$$\frac{\partial}{\partial\eta}=\begin{pmatrix}\frac{\partial}{\partial\eta_{11}} & \cdots & \frac{\partial}{\partial\eta_{m1}}\\ \vdots & \ddots & \vdots\\ \frac{\partial}{\partial\eta_{1n}} & \cdots & \frac{\partial}{\partial\eta_{mn}}\end{pmatrix},\qquad \frac{\partial}{\partial\overline{\eta}}=\begin{pmatrix}\frac{\partial}{\partial\overline{\eta}_{11}} & \cdots & \frac{\partial}{\partial\overline{\eta}_{m1}}\\ \vdots & \ddots & \vdots\\ \frac{\partial}{\partial\overline{\eta}_{1n}} & \cdots & \frac{\partial}{\partial\overline{\eta}_{mn}}\end{pmatrix}.$$

We can identify an element $g=(M,(\lambda,\mu;\kappa))$ of G^J, $M=\begin{pmatrix} A & B \\ C & D\end{pmatrix}\in Sp(n,\mathbb{R})$ with the element

$$\begin{pmatrix} A & 0 & B & A\,{}^t\mu - B\,{}^t\lambda \\ \lambda & I_m & \mu & \kappa \\ C & 0 & D & C\,{}^t\mu - D\,{}^t\lambda \\ 0 & 0 & 0 & I_m \end{pmatrix}$$

of $Sp(m+n,\mathbb{R})$.

We set

$$T_* = \frac{1}{\sqrt{2}}\begin{pmatrix} I_{m+n} & I_{m+n} \\ iI_{m+n} & -iI_{m+n}\end{pmatrix}.$$

We now consider the group G^J_* defined by

$$G^J_* := T_*^{-1} G^J T_*.$$

If $g=(M,(\lambda,\mu;\kappa))\in G^J$ with $M=\begin{pmatrix} A & B \\ C & D\end{pmatrix}\in Sp(n,\mathbb{R})$, then $T_*^{-1}gT_*$ is given by

$$T_*^{-1}gT_* = \begin{pmatrix} P_* & Q_* \\ \overline{Q}_* & \overline{P}_*\end{pmatrix},$$

where

$$P_* = \begin{pmatrix} P & \frac{1}{2}\left\{Q\ {}^t(\lambda+i\mu) - P\ {}^t(\lambda-i\mu)\right\} \\ \frac{1}{2}(\lambda+i\mu) & I_h + i\frac{\kappa}{2}\end{pmatrix},$$

$$Q_* = \begin{pmatrix} Q & \frac{1}{2}\left\{P\ {}^t(\lambda-i\mu) - Q\ {}^t(\lambda+i\mu)\right\} \\ \frac{1}{2}(\lambda-i\mu) & -i\frac{\kappa}{2}\end{pmatrix},$$

and P, Q are given by the formulas

$$P = \frac{1}{2}\ \{(A+D) + i\,(B-C)\} \tag{4.4}$$

and

$$Q = \frac{1}{2}\ \{(A-D) - i\,(B+C)\}. \tag{4.5}$$

From now on, we write

$$\left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P}\end{pmatrix}, \left(\frac{1}{2}(\lambda+i\mu),\ \frac{1}{2}(\lambda-i\mu);\ -i\frac{\kappa}{2}\right)\right) := \begin{pmatrix} P_* & Q_* \\ \overline{Q}_* & \overline{P}_*\end{pmatrix}.$$

In other words, we have the relation

$$T_*^{-1}\left(\begin{pmatrix} A & B \\ C & D\end{pmatrix}, (\lambda,\mu;\kappa)\right) T_* = \left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P}\end{pmatrix}, \left(\frac{1}{2}(\lambda+i\mu),\ \frac{1}{2}(\lambda-i\mu);\ -i\frac{\kappa}{2}\right)\right).$$

Let

$$H^{(n,m)}_{\mathbb{C}} := \left\{ (\xi,\eta\,;\zeta)\mid \xi,\eta\in\mathbb{C}^{(m,n)},\ \zeta\in\mathbb{C}^{(m,m)},\ \zeta+\eta\,{}^t\xi \text{ symmetric}\right\}$$

be the complex Heisenberg group endowed with the following multiplication

$$(\xi,\eta\,;\zeta)\circ(\xi',\eta';\zeta') := (\xi+\xi',\eta+\eta'\,;\zeta+\zeta'+\xi\,{}^t\eta' - \eta\,{}^t\xi')).$$

We define the semidirect product

$$SL(2n,\mathbb{C}) \ltimes H_{\mathbb{C}}^{(n,m)}$$

endowed with the following multiplication

$$\begin{aligned}&\left(\begin{pmatrix} P & Q \\ R & S \end{pmatrix}, (\xi,\eta\,;\zeta)\right)\cdot\left(\begin{pmatrix} P' & Q' \\ R' & S' \end{pmatrix}, (\xi',\eta';\zeta')\right)\\ &\quad=\left(\begin{pmatrix} P & Q \\ R & S \end{pmatrix}\begin{pmatrix} P' & Q' \\ R' & S' \end{pmatrix}, (\tilde{\xi}+\xi',\tilde{\eta}+\eta';\zeta+\zeta'+\tilde{\xi}\,{}^t\eta'-\tilde{\eta}\,{}^t\xi')\right),\end{aligned}$$

where $\tilde{\xi}=\xi P'+\eta R'$ and $\tilde{\eta}=\xi Q'+\eta S'$.

If we identify $H_{\mathbb{R}}^{(n,m)}$ with the subgroup

$$\left\{(\xi,\overline{\xi};i\kappa)\mid \xi\in\mathbb{C}^{(m,n)},\ \kappa\in\mathbb{R}^{(m,m)}\right\}$$

of $H_{\mathbb{C}}^{(n,m)}$, we have the following inclusion

$$G_*^J \subset SU(n,n)\ltimes H_{\mathbb{R}}^{(n,m)} \subset SL(2n,\mathbb{C})\ltimes H_{\mathbb{C}}^{(n,m)}.$$

We define the mapping $\Theta : G^J \longrightarrow G_*^J$ by

$$\Theta\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, (\lambda,\mu;\kappa)\right)=\left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, \left(\frac{1}{2}(\lambda+i\mu),\ \frac{1}{2}(\lambda-i\mu);\ -i\frac{\kappa}{2}\right)\right), \tag{4.6}$$

where P and Q are given by (4.4) and (4.5). We can see that if $g_1,g_2\in G^J$, then $\Theta(g_1g_2)=\Theta(g_1)\Theta(g_2)$.

According to [69, p. 250], G_*^J is of the Harish-Chandra type (cf. [53, p. 118]). Let

$$g_*=\left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, (\lambda,\mu;\,\kappa)\right)$$

be an element of G_*^J. Since the Harish-Chandra decomposition of an element $\begin{pmatrix} P & Q \\ R & S \end{pmatrix}$ in $SU(n,n)$ is given by

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix}=\begin{pmatrix} I_n & QS^{-1} \\ 0 & I_n \end{pmatrix}\begin{pmatrix} P-QS^{-1}R & 0 \\ 0 & S \end{pmatrix}\begin{pmatrix} I_n & 0 \\ S^{-1}R & I_n \end{pmatrix},$$

the P_*^+-component of the following element

$$g_*\cdot\left(\begin{pmatrix} I_n & W \\ 0 & I_n \end{pmatrix}, (0,\eta;0)\right),\quad W\in\mathbb{D}_n$$

of $SL(2n,\mathbb{C})\ltimes H_{\mathbb{C}}^{(n,m)}$ is given by

$$\left(\begin{pmatrix} I_n & (PW+Q)(\overline{Q}W+\overline{P})^{-1} \\ 0 & I_n \end{pmatrix}, \left(0,\ (\eta+\lambda W+\mu)(\overline{Q}W+\overline{P})^{-1}\,;0\right)\right). \tag{4.7}$$

We can identify $\mathbb{D}_{n,m}$ with the subset

$$\left\{\left(\begin{pmatrix} I_n & W \\ 0 & I_n \end{pmatrix}, (0,\eta;0)\right)\ \Big|\ W\in\mathbb{D}_n,\ \eta\in\mathbb{C}^{(m,n)}\right\}$$

of the complexification of G^J_*. Indeed, $\mathbb{D}_{n,m}$ is embedded into P^+_* given by

$$P^+_* = \left\{ \left(\begin{pmatrix} I_n & W \\ 0 & I_n \end{pmatrix}, (0,\eta;0) \right) \,\Big|\, W = {}^tW \in \mathbb{C}^{(n,n)},\ \eta \in \mathbb{C}^{(m,n)} \right\}.$$

This is a generalization of the Harish-Chandra embedding (cf. [53, p. 119]). Then we get the *natural transitive action* of G^J_* on $\mathbb{D}_{n,m}$ defined by

$$\begin{aligned} &\left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, (\xi, \overline{\xi};\, i\kappa) \right) \cdot (W,\eta) \\ &\quad = \Big((PW+Q)(\overline{Q}W+\overline{P})^{-1}, (\eta+\xi W+\overline{\xi})(\overline{Q}W+\overline{P})^{-1} \Big), \end{aligned} \tag{4.8}$$

where $\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \in G_*,\ \xi \in \mathbb{C}^{(m,n)},\ \kappa \in \mathbb{R}^{(m,m)}$ and $(W,\eta) \in \mathbb{D}_{n,m}$.

The author [72] proved that the action (1.2) of G^J on $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$ is compatible with the action (4.8) of G^J_* on $\mathbb{D}_{n,m}$ through the *partial Cayley transform* $\Psi : \mathbb{D}_{n,m} \longrightarrow \mathbb{H}_{n,m}$ defined by

$$\Psi(W,\eta) := \Big(i(I_n+W)(I_n-W)^{-1},\, 2\,i\,\eta\,(I_n-W)^{-1} \Big). \tag{4.9}$$

In other words, if $g_0 \in G^J$ and $(W,\eta) \in \mathbb{D}_{n,m}$,

$$g_0 \cdot \Psi(W,\eta) = \Psi(g_* \cdot (W,\eta)), \tag{4.10}$$

where $g_* = T_*^{-1} g_0 T_*$. Ψ is a biholomorphic mapping of $\mathbb{D}_{n,m}$ onto $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$ which gives the partially bounded realization of $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$ by $\mathbb{D}_{n,m}$. The inverse of Ψ is

$$\Psi^{-1}(\Omega, Z) = \Big((\Omega - iI_n)(\Omega + iI_n)^{-1},\, Z(\Omega+iI_n)^{-1} \Big). \tag{4.11}$$

5. Invariant metrics and Laplacians on the Siegel–Jacobi disk

For $W = (w_{ij}) \in \mathbb{D}_n$, we write $dW = (dw_{ij})$ and $d\overline{W} = (d\overline{w}_{ij})$. We put

$$\frac{\partial}{\partial W} = \left(\frac{1+\delta_{ij}}{2}\,\frac{\partial}{\partial w_{ij}} \right) \qquad \text{and} \qquad \frac{\partial}{\partial \overline{W}} = \left(\frac{1+\delta_{ij}}{2}\,\frac{\partial}{\partial \overline{w}_{ij}} \right).$$

Using the Cayley transform $\Psi : \mathbb{D}_n \longrightarrow \mathbb{H}_n$, Siegel [57] showed that

$$ds^2_* = 4\,\sigma\Big((I_n - W\overline{W})^{-1} dW\, (I_n - \overline{W}W)^{-1} d\overline{W} \Big) \tag{5.1}$$

is a G_*-invariant Riemannian metric on $\mathbb{D}_n$ and Maass [37] showed that its Laplacian is given by

$$\Delta_* = \sigma\left((I_n - W\overline{W})\, {}^t\!\left((I_n - W\overline{W}) \frac{\partial}{\partial \overline{W}} \right) \frac{\partial}{\partial W} \right). \tag{5.2}$$

Yang [73] proved the following theorems.

Theorem 5.1. *For any two positive real numbers A and B, the following metric $d\tilde{s}^2_{n,m;A,B}$ defined by*

$$\begin{aligned}
ds^2_{\mathbb{D}_{n,m};A,B} = {} & 4\,A\,\sigma\Big((I_n - W\overline{W})^{-1}dW(I_n-\overline{W}W)^{-1}d\overline{W}\Big)\\
& + 4\,B\,\Big\{\sigma\Big((I_n - W\overline{W})^{-1}\,{}^t(d\eta)\,\beta\Big)\\
& + \sigma\Big((\eta\overline{W}-\overline{\eta})(I_n - W\overline{W})^{-1}dW(I_n-\overline{W}W)^{-1}\,{}^t(d\overline{\eta})\Big)\\
& + \sigma\Big((\overline{\eta}W-\eta)(I_n - \overline{W}W)^{-1}d\overline{W}(I_n-W\overline{W})^{-1}\,{}^t(d\eta)\Big)\\
& - \sigma\Big((I_n - W\overline{W})^{-1}\,{}^t\eta\,\eta\,(I_n-\overline{W}W)^{-1}\overline{W}dW(I_n-\overline{W}W)^{-1}d\overline{W}\Big)\\
& - \sigma\Big(W(I_n - \overline{W}W)^{-1}\,{}^t\overline{\eta}\,\overline{\eta}\,(I_n-W\overline{W})^{-1}dW(I_n-\overline{W}W)^{-1}d\overline{W}\Big)\\
& + \sigma\Big((I_n - W\overline{W})^{-1\,t}\eta\,\overline{\eta}\,(I_n-W\overline{W})^{-1}dW(I_n-\overline{W}W)^{-1}d\overline{W}\Big)\\
& + \sigma\Big((I_n - \overline{W})^{-1}\,{}^t\overline{\eta}\,\eta\,\overline{W}\,(I_n-W\overline{W})^{-1}dW(I_n-\overline{W}W)^{-1}d\overline{W}\Big)\\
& + \sigma\Big((I_n - \overline{W})^{-1}(I_n-W)(I_n-\overline{W}W)^{-1}\,{}^t\overline{\eta}\,\eta\,(I_n-\overline{W}W)^{-1}\\
& \qquad\times\,(I_n-\overline{W})(I_n-W)^{-1}dW(I_n-\overline{W}W)^{-1}d\overline{W}\Big)\\
& - \sigma\Big((I_n - W\overline{W})^{-1}(I_n-W)(I_n-\overline{W})^{-1}\,{}^t\overline{\eta}\,\eta\,(I_n-W)^{-1}\\
& \qquad\times\,dW(I_n-\overline{W}W)^{-1}d\overline{W}\Big)\Big\}
\end{aligned}$$

is a Riemannian metric on $\mathbb{D}_{n,m}$ which is invariant under the action (4.8) *of the Jacobi group G^J_*.*

Proof. See Theorem 1.3 in [73]. □

Theorem 5.2. *The following differential operators $\mathbb{S}_1$ and $\mathbb{S}_2$ on $\mathbb{D}_{n,m}$ defined by*

$$\mathbb{S}_1 = \sigma\left((I_n-\overline{W}W)\frac{\partial}{\partial\eta}\,{}^t\!\left(\frac{\partial}{\partial\overline{\eta}}\right)\right)$$

and

$$\begin{aligned}
\mathbb{S}_2 = {} & \sigma\left((I_n-W\overline{W})\,{}^t\!\left((I_n-W\overline{W})\frac{\partial}{\partial\overline{W}}\right)\frac{\partial}{\partial W}\right)\\
& + \sigma\left({}^t(\eta-\overline{\eta}\,W)\,{}^t\!\left(\frac{\partial}{\partial\overline{\eta}}\right)(I_n-\overline{W}W)\frac{\partial}{\partial W}\right)\\
& + \sigma\left((\overline{\eta}-\eta\,\overline{W})\,{}^t\!\left((I_n-W\overline{W})\frac{\partial}{\partial\overline{W}}\right)\frac{\partial}{\partial\eta}\right)\\
& - \sigma\left(\eta\overline{W}(I_n-W\overline{W})^{-1}\,{}^t\eta\,{}^t\!\left(\frac{\partial}{\partial\overline{\eta}}\right)(I_n-\overline{W}W)\frac{\partial}{\partial\eta}\right)
\end{aligned}$$

$$- \sigma \left(\overline{\eta} W (I_n - \overline{W} W)^{-1}\, {}^t\overline{\eta}\, {}^t\!\left(\frac{\partial}{\partial \overline{\eta}}\right) (I_n - \overline{W} W) \frac{\partial}{\partial \eta} \right)$$
$$+ \sigma \left(\overline{\eta} (I_n - W \overline{W})^{-1}\, {}^t\eta\, {}^t\!\left(\frac{\partial}{\partial \overline{\eta}}\right) (I_n - \overline{W} W) \frac{\partial}{\partial \eta} \right)$$
$$+ \sigma \left(\eta \overline{W} W (I_n - \overline{W} W)^{-1}\, {}^t\overline{\eta}\, {}^t\!\left(\frac{\partial}{\partial \overline{\eta}}\right) (I_n - \overline{W} W) \frac{\partial}{\partial \eta} \right)$$

are invariant under the action (4.8) *of* G^J_*. *The following differential operator*

$$\Delta_{\mathbb{D}_{n,m};A,B} := \frac{1}{A}\, \mathbb{S}_2 + \frac{1}{B}\, \mathbb{S}_1 \tag{5.3}$$

is the Laplacian of the invariant metric $ds^2_{\mathbb{D}_{n,m};A,B}$ *on* $\mathbb{D}_{n,m}$.

Proof. See Theorem 1.4 in [73]. □

Itoh, Ochiai and Yang [26] proved that the following differential operator on $\mathbb{D}_{n,m}$ defined by

$$\mathbb{S}_3 = \det(I_n - \overline{W} W)\, \det \left(\frac{\partial}{\partial \eta}\, {}^t\!\left(\frac{\partial}{\partial \overline{\eta}} \right) \right)$$

is invariant under the action (4.8) of G^J_* on $\mathbb{D}_{n,m}$. Furthermore the authors [26] proved that the following matrix-valued differential operator on $\mathbb{D}_{n,m}$ defined by

$$\mathbb{J} := {}^t\!\left(\frac{\partial}{\partial \overline{\eta}} \right) (I_n - \overline{W} W) \frac{\partial}{\partial \eta}$$

and each (k, l)-entry $\mathbb{J}_{kl}$ of $\mathbb{J}$ given by

$$\mathbb{J}_{kl} = \sum_{i,j=1}^{n} \left(\delta_{ij} - \sum_{r=1}^{n} \overline{w}_{ir}\, w_{jr} \right) \frac{\partial^2}{\partial \overline{\eta}_{ki} \partial \eta_{lj}}, \quad 1 \le k, l \le m$$

are invariant under the action (4.8) of G^J_* on $\mathbb{D}_{n,m}$.

$$\mathbb{S}_* = [\mathbb{S}_1, \mathbb{S}_2] = \mathbb{S}_1 \mathbb{S}_2 - \mathbb{S}_2 \mathbb{S}_1$$

is an invariant differential operator of degree three on $\mathbb{D}_{n,m}$ and

$$\mathbb{Q}_{kl} = [\mathbb{S}_3, \mathbb{J}_{kl}] = \mathbb{S}_3 \mathbb{J}_{kl} - \mathbb{J}_{kl} \mathbb{S}_3, \quad 1 \le k, l \le m$$

is an invariant differential operator of degree $2n + 1$ on $\mathbb{D}_{n,m}$.

Indeed it is very complicated and difficult at this moment to express the generators of the algebra of all G^J_*-invariant differential operators on $\mathbb{D}_{n,m}$ explicitly.

6. A fundamental domain for the Siegel–Jacobi space

Let

$$\mathscr{P}_n = \left\{ Y \in \mathbb{R}^{(n,n)} \mid Y = {}^tY > 0 \right\}$$

be an open connected cone in $\mathbb{R}^N$ with $N = n(n+1)/2$. Then the general linear group $GL(n, \mathbb{R})$ acts on $\mathscr{P}_n$ transitively by

$$g \circ Y := gY\,{}^tg, \quad g \in GL(n, \mathbb{R}),\ Y \in \mathscr{P}_g. \tag{6.1}$$

Thus $\mathscr{P}_n$ is a symmetric space diffeomorphic to $GL(n, \mathbb{R})/O(n)$.

The fundamental domain $\mathscr{R}_n$ for $GL(n, \mathbb{Z})\backslash\mathscr{P}_n$ which was found by H. Minokwski [42] is defined as a subset of $\mathscr{P}_n$ consisting of $Y = (y_{ij}) \in \mathscr{P}_n$ satisfying the following conditions (M.1) and (M.2):

(M.1) $aY\,{}^ta \geq y_{kk}$ for every $a = (a_i) \in \mathbb{Z}^n$ in which $a_k, \ldots, a_n$ are relatively prime for $k = 1, 2, \ldots, n$.

(M.2) $y_{k,k+1} \geq 0$ for $k = 1, \ldots, n-1$.

We say that a point of $\mathscr{R}_n$ is *Minkowski reduced*.

Let $\Gamma_n = Sp(n, \mathbb{Z})$ be the Siegel modular group of degree n. Siegel determined a fundamental domain $\mathscr{F}_n$ for $\Gamma_n\backslash\mathbb{H}_n$. We say that $\Omega = X + iY \in \mathbb{H}_n$ with X, Y real is *Siegel reduced* or *S-reduced* if it has the following three properties:

(S.1) $\det(\mathrm{Im}(\gamma \cdot \Omega)) \leq \det(\mathrm{Im}\,(\Omega))$ for all $\gamma \in \Gamma_n$;

(S.2) $Y = \mathrm{Im}(\Omega)$ is *Minkowski reduced*, that is, $Y \in \mathscr{R}_n$;

(S.3) $|x_{ij}| \leq \frac{1}{2}$ for $1 \leq i, j \leq n$, where $X = (x_{ij})$.

$\mathscr{F}_n$ is defined as the set of all Siegel reduced points in $\mathbb{H}_n$. Using the highest point method, Siegel proved the following (F1)–(F3):

(F1) $\Gamma_n \cdot \mathscr{F}_n = \mathbb{H}_n$, i.e., $\mathbb{H}_n = \bigcup_{\gamma \in \Gamma_n} \gamma \cdot \mathscr{F}_n$;

(F2) $\mathscr{F}_n$ is closed in $\mathbb{H}_n$;

(F3) $\mathscr{F}_n$ is connected and the boundary of $\mathscr{F}_n$ consists of a finite number of hyperplanes.

Let E_{kj} be the $m \times n$ matrix with entry 1 where the kth row and the $j$the column meet, and all other entries 0. For an element $\Omega \in \mathbb{H}_n$, we set for brevity

$$F_{kj}(\Omega) := E_{kj}\,\Omega, \qquad 1 \leq k \leq m,\ 1 \leq j \leq n.$$

For each $\Omega \in \mathscr{F}_n$, we define the subset P_Ω of $\mathbb{C}^{(m,n)}$ by

$$P_\Omega = \left\{ \sum_{k=1}^{m} \sum_{j=1}^{n} \lambda_{kj} E_{kj} + \sum_{k=1}^{m} \sum_{j=1}^{n} \mu_{kj} F_{kj}(\Omega) \,\Big|\, 0 \leq \lambda_{kj}, \mu_{kj} \leq 1 \right\}.$$

For each $\Omega \in \mathscr{F}_n$, we define the subset D_Ω of $\mathbb{H}_{n,m}$ by

$$D_\Omega = \left\{ (\Omega, Z) \in \mathbb{H}_{n,m} \mid Z \in P_\Omega \right\}.$$

Let

$$\Gamma_{n,m} = \Gamma_n \ltimes H_{\mathbb{Z}}^{(n,m)} \tag{6.2}$$

be the Siegel–Jacobi (or simply Jacobi) modular group of degree n and index m.

Yang found a fundamental domain $\mathscr{F}_{n,m}$ for $\Gamma_{n,m}\backslash\mathbb{H}_{n,m}$ using Siegel's fundamental domain $\mathscr{F}_n$ in [70].

Theorem 6.1. *The set*

$$\mathscr{F}_{n,m} := \bigcup_{\Omega\in\mathscr{F}_n} D_\Omega$$

is a fundamental domain for $\Gamma_{n,m}\backslash\mathbb{H}_{n,m}$.

Proof. See Theorem 3.1 in [70]. □

7. Jacobi forms

Let ρ be a rational representation of $GL(n,\mathbb{C})$ on a finite-dimensional complex vector space V_ρ. Let $\mathcal{M}\in\mathbb{R}^{(m,m)}$ be a symmetric half-integral semi-positive definite matrix of degree m. The canonical automorphic factor

$$J_{\rho,\mathcal{M}} : G^J\times\mathbb{H}_{n,m}\longrightarrow GL(V_\rho)$$

for G^J on $\mathbb{H}_{n,m}$ is given as follows:

$$J_{\rho,\mathcal{M}}((g,(\lambda,\mu;\kappa)),(\Omega,Z)) = e^{2\,\pi\, i\,\sigma\left(\mathcal{M}(Z+\lambda\,\Omega+\mu)(C\Omega+D)^{-1}C\,{}^t(Z+\lambda\,\Omega\,+\,\mu)\right)} \times\, e^{-2\,\pi\, i\,\sigma\left(\mathcal{M}(\lambda\,\Omega\,{}^t\lambda\,+\,2\,\lambda\,{}^tZ+\,\kappa+\mu\,{}^t\lambda)\right)}\rho(C\,\Omega+D),$$

where $g=\left(\begin{smallmatrix} A & B\\ C & D\end{smallmatrix}\right)\in Sp(n,\mathbb{R}),\ (\lambda,\mu;\kappa)\in H_{\mathbb{R}}^{(n,m)}$ and $(\Omega,Z)\in\mathbb{H}_{n,m}$. We refer to [66] for a geometrical construction of $J_{\rho,\mathcal{M}}$.

Let $C^\infty(\mathbb{H}_{n,m},V_\rho)$ be the algebra of all C^∞ functions on $\mathbb{H}_{n,m}$ with values in V_ρ. For $f\in C^\infty(\mathbb{H}_{n,m},V_\rho)$, we define

$$(f|_{\rho,\mathcal{M}}[(g,(\lambda,\mu;\kappa))])\,(\Omega,Z) = J_{\rho,\mathcal{M}}((g,(\lambda,\mu;\kappa)),(\Omega,Z))^{-1}\, f\left(g\cdot\Omega,(Z+\lambda\,\Omega+\mu)(C\,\Omega+D)^{-1}\right), \tag{7.1}$$

where $g=\left(\begin{smallmatrix} A & B\\ C & D\end{smallmatrix}\right)\in Sp(n,\mathbb{R}),\ (\lambda,\mu;\kappa)\in H_{\mathbb{R}}^{(n,m)}$ and $(\Omega,Z)\in\mathbb{H}_{n,m}$.

Definition 7.1. Let ρ and $\mathcal{M}$ be as above. Let

$$H_{\mathbb{Z}}^{(n,m)} := \left\{(\lambda,\mu;\kappa)\in H_{\mathbb{R}}^{(n,m)}\mid\ \lambda,\mu,\kappa \text{ integral }\right\}$$

be the discrete subgroup of $H_{\mathbb{R}}^{(n,m)}$. A Jacobi form of index $\mathcal{M}$ with respect to ρ on a subgroup Γ of Γ_n of finite index is a holomorphic function $f\in C^\infty(\mathbb{H}_{n,m},V_\rho)$ satisfying the following conditions (A) and (B):

(A) $f|_{\rho,\mathcal{M}}[\tilde\gamma]=f$ for all $\tilde\gamma\in\widetilde\Gamma:=\Gamma\ltimes H_{\mathbb{Z}}^{(n,m)}$.

(B) For each $M\in\Gamma_n,\ f|_{\rho,\mathcal{M}}[M]$ has a Fourier expansion of the following form:

$$(f|_{\rho,\mathcal{M}}[M])(\Omega,Z) = \sum_{\substack{T={}^tT\geq 0\\ \text{half-integral}}}\ \sum_{R\in\mathbb{Z}^{(n,m)}} c(T,R)\cdot e^{\frac{2\pi i}{\lambda_\Gamma}\,\sigma(T\Omega)}\cdot e^{2\pi i\,\sigma(RZ)}$$

with $\lambda_\Gamma(\neq 0)\in\mathbb{Z}$ and $c(T,R)\neq 0$ only if $\begin{pmatrix}\frac{1}{\lambda_\Gamma}T & \frac12 R\\ \frac12\,{}^tR & \mathcal{M}\end{pmatrix}\geq 0.$

If $n \geq 2$, the condition (B) is superfluous by the Köcher principle (cf. [82] Lemma 1.6). We denote by $J_{\rho,\mathcal{M}}(\Gamma)$ the vector space of all Jacobi forms of index $\mathcal{M}$ with respect to ρ on Γ. Ziegler (cf. [82] Theorem 1.8 or [12] Theorem 1.1) proves that the vector space $J_{\rho,\mathcal{M}}(\Gamma)$ is finite dimensional. In the special case $\rho(A) = (\det(A))^k$ with $A \in GL(n,\mathbb{C})$ and a fixed $k \in \mathbb{Z}$, we write $J_{k,\mathcal{M}}(\Gamma)$ instead of $J_{\rho,\mathcal{M}}(\Gamma)$ and call k the *weight* of the corresponding Jacobi forms. For more results about Jacobi forms with $n > 1$ and $m > 1$, we refer to [62]–[68] and [82]. Jacobi forms play an important role in lifting elliptic cusp forms to Siegel cusp forms of degree $2n$ (cf. [24, 25]).

Now we will make brief historical remarks on Jacobi forms. In 1985, the names Jacobi group and Jacobi forms got kind of standard by the classic book [12] by Eichler and Zagier to remind of Jacobi's "Fundamenta nova theoriae functionum ellipticorum", which appeared in 1829 (cf. [27]). Before [12] these objects appeared more or less explicitly and under different names in the work of many authors. In 1966 Pyatetski–Shapiro [49] discussed the Fourier–Jacobi expansion of Siegel modular forms and the field of modular abelian functions. He gave the dimension of this field in the higher degree. About the same time Satake [52]–[53] introduced the notion of "groups of Harish-Chandra type" which are non reductive but still behave well enough so that he could determine their canonical automorphic factors and kernel functions. Shimura [54]–[55] gave a new foundation of the theory of complex multiplication of Abelian functions using Jacobi theta functions. Kuznetsov [34] constructed functions which are almost Jacobi forms from ordinary elliptic modular functions. Starting 1981, Berndt [3]–[5] published some papers which studied the field of arithmetic Jacobi functions, ending up with a proof of Shimura reciprocity law for the field of these functions with arbitrary level. Furthermore he investigated the discrete series for the Jacobi group G^J and developed the spectral theory for $L^2(\Gamma_{n,m}\backslash G^J)$ in the case $n = m = 1$ (cf. [6]–[8]). The connection of Jacobi forms to modular forms was given by Maass, Andrianov, Kohnen, Shimura, Eichler and Zagier. This connection is pictured as follows. For k even, we have the following isomorphisms

$$M_k^*(\Gamma_2) \cong J_{k,1}(\Gamma_1) \cong M^+_{k-\frac{1}{2}}\left(\Gamma_0^{(1)}(4)\right) \cong M_{2k-2}(\Gamma_1). \tag{7.2}$$

Here $M_k^*(\Gamma_2)$ denotes Maass' Spezialschar or Maass space and $M^+_{k-\frac{1}{2}}\left(\Gamma_0^{(1)}(4)\right)$ denotes the Kohnen plus space. For a precise detail, we refer to [39]–[41], [1], [12], [29, 30] and [80]. In 1982 Tai [60] gave asymptotic dimension formulae for certain spaces of Jacobi forms for arbitrary n and $m = 1$ and used these ones to show that the moduli $\mathcal{A}_n$ of principally polarized Abelian varieties of dimension n is *of general type* for $n \geq 9$. Feingold and Frenkel [13] essentially discussed Jacobi forms in the context of Kac–Moody Lie algebras generalizing the Maass correspondence to higher level. Gritsenko [17] studied Fourier–Jacobi expansions and a non-commutative Hecke ring in connection with the Jacobi group. After 1985 the theory of Jacobi forms for $n = m = 1$ had been studied more or less systematically by the Zagier school. A large part of the theory of Jacobi forms of higher degree was investigated by Kramer [32, 33], Runge [51], Yang [62]–[66]and Ziegler [82]. There were several attempts to establish L-functions in the context of the Jacobi group by Murase [46, 47] and Sugano [48] using the so-called "Whittaker–Shintani functions". Kramer [32, 33] developed an arithmetic theory of Jacobi forms of higher degree. Runge [51] discussed some part of the geometry of Jacobi forms for arbitrary n and $m = 1$. For a good survey on some motivation and background for the study of Jacobi forms, we refer to [9]. The theory of Jacobi forms has been extensively studied by many people until now and has many applications in other areas like geometry and physics.

8. Singular Jacobi forms

Definition 8.1. A Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma_n)$ is said to be cuspidal if $\begin{pmatrix} T & \frac{1}{2}R \\ \frac{1}{2}\,{}^tR & \mathcal{M} \end{pmatrix} > 0$ for any T, R with $c(T,R) \neq 0$. A Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma_n)$ is said to be singular if it admits a Fourier expansion such that a Fourier coefficient $c(T,R)$ vanishes unless $\det \begin{pmatrix} T & \frac{1}{2}R \\ \frac{1}{2}\,{}^tR & \mathcal{M} \end{pmatrix} = 0$.

Let $\mathscr{P}_{n,m} = \mathscr{P}_n \times \mathbb{R}^{(m,n)}$ be the Minkowski–Euclid space, where $\mathscr{P}_n$ is the open cone consisting of positive symmetric $n \times n$ real matrices. For a variable $(Y,V) \in \mathscr{P}_{n,m}$ with $Y \in \mathscr{P}_n$ and $V \in \mathbb{R}^{(m,n)}$, we put

$$Y = (y_{\mu\nu}) \ \text{ with } \ y_{\mu\nu} = y_{\nu\mu}, \qquad V = (v_{kl}),$$
$$\frac{\partial}{\partial Y} = \left(\frac{1+\delta_{\mu\nu}}{2} \frac{\partial}{\partial y_{\mu\nu}} \right), \qquad \frac{\partial}{\partial V} = \left(\frac{\partial}{\partial v_{kl}} \right),$$

where $1 \leq \mu, \nu$, $l \leq n$ and $1 \leq k \leq m$.

We define the following differential operator

$$M_{n,m,\mathcal{M}} := \det(Y) \cdot \det \left(\frac{\partial}{\partial Y} + \frac{1}{8\,\pi}\, {}^t\!\left(\frac{\partial}{\partial V} \right) \mathcal{M}^{-1} \frac{\partial}{\partial V} \right). \tag{8.1}$$

In [65], Yang characterized singular Jacobi forms in the following way:

Theorem 8.1. *Let $f \in J_{\rho,\mathcal{M}}(\Gamma_n)$ be a Jacobi form of index $\mathcal{M}$ with respect to a rational representation ρ of $GL(n,\mathbb{C})$. Then the following conditions are equivalent:*

(Sing-1) *f is a singular Jacobi form.*
(Sing-2) *f satisfies the differential equation $M_{n,m,\mathcal{M}} f = 0$.*

Proof. See Theorem 4.1 in [65]. □

Theorem 8.2. *Let $2\mathcal{M}$ be a symmetric, positive definite, unimodular even matrix of degree m. Assume that ρ is irreducible and satisfies the condition*

$$\rho(A) = \rho(-A) \quad \text{for all } A \in GL(n, \mathbb{C}).$$

Then a nonvanishing Jacobi form in $J_{\rho,\mathcal{M}}(\Gamma_n)$ is singular if and only if $2\,k(\rho) < n + m$.

Proof. See Theorem 4.5 in [65]. □

Remark 8.1. We let

$$GL_{n,m} := GL(n, \mathbb{R}) \ltimes \mathbb{R}^{(m,n)}$$

be the semidirect product of $GL(n, \mathbb{R})$ and $\mathbb{R}^{(m,n)}$ with multiplication law

$$(A, a) \cdot (B, b) := (AB, a\,{}^tB^{-1} + b), \qquad A, B \in GL(n, \mathbb{R}), \quad a, b \in \mathbb{R}^{(m,n)}.$$

Then we have the *natural action* of $GL_{n,m}$ on the Minkowski–Euclid space $\mathscr{P}_{n,m}$ defined by

$$(A, a) \cdot (Y, \zeta) := \big(AY\,{}^tA,\ (\zeta + a)\,{}^tA\big), \tag{8.2}$$

where $(A, a) \in GL_{n,m},\ Y \in \mathscr{P}_n,\ \zeta \in \mathbb{R}^{(m,n)}$. Without difficulty we see that the differential operator $M_{n,m,\mathcal{M}}$ is invariant under the action (8.2) of $GL_{n,m}$. We refer to [77] for more detail about invariant differential operators on the Minkowski–Euclid space $\mathscr{P}_{n,m}$.

9. The Siegel–Jacobi operator

Let ρ be a rational representation of $GL(n, \mathbb{C})$ on a finite-dimensional vector space V_ρ. For a positive integer $r < n$, we let $\rho^{(r)} : GL(r, \mathbb{C}) \longrightarrow GL(V_\rho)$ be a rational representation of $GL(r, \mathbb{C})$ defined by

$$\rho^{(r)}(a)v := \rho\left(\begin{pmatrix} a & 0 \\ 0 & iI_{n-r} \end{pmatrix}\right) v, \quad a \in GL(r, \mathbb{C}),\ v \in V_\rho.$$

The Siegel–Jacobi operator $\Psi_{n,r} : J_{\rho,\mathcal{M}}(\Gamma_n) \longrightarrow J_{\rho^{(r)},\mathcal{M}}(\Gamma_n)$ is defined by

$$(\Psi_{n,r} f)(\Omega, Z) := \lim_{t \longrightarrow \infty} f\left(\begin{pmatrix} \Omega & 0 \\ 0 & i\,tI_{n-r} \end{pmatrix}, (Z, 0)\right),$$

where $f \in J_{\rho,\mathcal{M}}(\Gamma_n),\ \Omega \in \mathbb{H}_r$ and $Z \in \mathbb{C}^{(m,r)}$.

In [62], Yang investigated the injectivity, surjectivity and bijectivity of the Siegel–Jacobi operator.

Theorem 9.1. *Let $2\mathcal{M}$ be a symmetric, positive definite, unimodular even matrix of degree m. Assume that ρ is irreducible and satisfies the condition*

$$\rho(A) = \rho(-A) \quad \text{for all } A \in GL(n, \mathbb{C}).$$

If $2\,k(\rho) < n + \operatorname{rank}(\mathcal{M})$, then the Siegel–Jacobi operator $\Psi_{n,n-1}$ is injective.

Proof. See Theorem 3.5 in [62]. □

Theorem 9.2. *Let $2\mathcal{M}$ be a symmetric, positive definite, unimodular even matrix of degree m. Assume that ρ is irreducible and satisfies the condition*

$$\rho(A) = \rho(-A) \quad \text{for all } A \in GL(n, \mathbb{C}).$$

If $2\,k(\rho) + 1 < n + \mathrm{rank}(\mathcal{M})$, then the Siegel–Jacobi operator $\Psi_{n,n-1}$ is an isomorphism.

Proof. See Theorem 3.6 in [62]. □

Theorem 9.3. *Let $2\mathcal{M}$ be a symmetric, positive definite, unimodular even matrix of degree m. Assume that $2\,k(\rho) > 4\,n + \mathrm{rank}(\mathcal{M})$ and $k \equiv 0 \pmod 2$. Then the Siegel–Jacobi operator $\Psi_{n,n-1}$ is an isomorphism.*

Proof. See Theorem 3.7 in [62]. □

Now we review the action of the Hecke operators on Jacobi forms. For a positive integer ℓ, we define

$$O_n(\ell) := \big\{ M \in \mathbb{Z}^{(2n,2n)} \mid {}^tMJ_nM = \ell J_n \big).$$

Then $O_n(\ell)$ is decomposed into finitely many double cosets *mod* Γ_n, that is,

$$O_n(\ell) = \bigcup_{j=1}^{s} \Gamma_n g_j \Gamma_n \qquad \text{(disjoint union)}.$$

We define

$$T(\ell) := \sum_{j=1}^{s} \Gamma_n g_j \Gamma_n \in \mathscr{H}^{(n)}, \qquad \text{the Hecke algebra.}$$

Let $M \in O_n(\ell)$. For a Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma_n)$, we define

$$f|_{\rho,\mathcal{M}}(\Gamma_n M \Gamma_n) := \ell^{nk(\rho) - \frac{n(n+1)}{2}} \sum_{j=1}^{s} f|_{\rho,\mathcal{M}}[(M_j, (0,0;0)))], \tag{9.1}$$

where $\Gamma_n M \Gamma_n = \bigcup_{j=1}^{s} \Gamma_n M_j$ (finite disjoint union) and $k(\rho)$ denotes the weight of ρ. We see easily that if $M \in O_n(\ell)$ and $f \in J_{\rho,\mathcal{M}}(\Gamma_n)$, then

$$f|_{\rho,\mathcal{M}}(\Gamma_n M \Gamma_n) \in J_{\rho,\ell\mathcal{M}}(\Gamma_n).$$

For a prime p, we define

$$O_{n,p} := \bigcup_{l=0}^{\infty} O_n(p^l).$$

Let $\check{\mathscr{L}}_{n,p}$ be the $\mathbb{C}$-module generated by all left cosets $\Gamma_n M,\ M \in O_{n,p}$ and $\check{\mathscr{H}}_{n,p}$ the $\mathbb{C}$-module generated by all double cosets $\Gamma_n M \Gamma_n,\ M \in O_{n,p}$. Then $\check{\mathscr{H}}_{n,p}$ is a commutative associative algebra. We associate to a double coset

$$\Gamma_n M \Gamma_n = \bigcup_{i=1}^{s} \Gamma_n M_i, \qquad M, M_i \in O_{n,p} \quad \text{(disjoint union)}$$

the element

$$j(\Gamma_n M \Gamma_n) = \sum_{i=1}^{s} \Gamma_n M_i \in \check{\mathscr{L}}_{n,p}.$$

We extend j linearly to the Hecke algebra $\check{\mathscr{H}}_{n,p}$ and then we have a monomorphism $j : \check{\mathscr{H}}_{n,p} \longrightarrow \check{\mathscr{L}}_{n,p}$. We now define a bilinear mapping

$$\check{\mathscr{H}}_{n,p} \times \check{\mathscr{L}}_{n,p} \longrightarrow \check{\mathscr{L}}_{n,p}$$

by

$$(\Gamma_n M \Gamma_n) \cdot (\Gamma_n M_0) = \sum_{i=1}^{s} \Gamma_n M_i M_0, \qquad \text{where } \Gamma_n M \Gamma_n = \bigcup_{i=1}^{s} \Gamma_n M_i.$$

This mapping is well defined because the definition does not depend on the choice of representatives.

Let $f \in J_{\rho,\mathcal{M}}(\Gamma_n)$ be a Jacobi form. For a left coset $L := \Gamma_n N$ with $N \in O_{n,p}$, we put

$$f|L := f|_{\rho,\mathcal{M}}[(N,(0,0;0))]. \tag{9.2}$$

We extend this operator (9.2) linearly to $\check{\mathscr{L}}_{n,p}$. If $T \in \check{\mathscr{H}}_{n,p}$, we write

$$f|T := f|j(T).$$

Obviously we have

$$(f|T)L = f|(TL), \qquad f \in J_{\rho,\mathcal{M}}(\Gamma_n).$$

In a left coset $\Gamma_n M,\ M \in O_{n,p}$, we can choose a representative M of the form

$$M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, \quad {}^tAD = p^{k_0} I_n, \qquad {}^tBD = {}^tDB,$$

$$A = \begin{pmatrix} a & {}^t\alpha \\ 0 & A^* \end{pmatrix}, \qquad B = \begin{pmatrix} b & {}^t\beta_1 \\ \beta_2 & B^* \end{pmatrix}, \quad \Delta = \begin{pmatrix} d & 0 \\ \delta & D^* \end{pmatrix},$$

where $\alpha, \beta_1, \beta_2, \delta \in \mathbb{Z}^{n-1}$. Then we have

$$M^* = \begin{pmatrix} A^* & B^* \\ 0 & D^* \end{pmatrix} \in O_{n-1,p}.$$

For an integer $r \in \mathbb{Z}$, we define

$$(\Gamma_n M)^* := \frac{1}{d^r} \Gamma_{n-1} M^*.$$

If $\Gamma_n M \Gamma_n = \bigcup_{j=1}^{s} \Gamma_n M_j$ (disjoint union), $M, M_j \in O_{n,p}$, then we define in a natural way

$$(\Gamma_n M \Gamma_n)^* := \frac{1}{d^r} \sum_{j=1}^{s} \Gamma_{n-1} M_j^*. \tag{9.3}$$

We extend the above map (9.3) linearly on $\check{\mathscr{H}}_{n,p}$ and then we have an algebra homomorphism

$$\check{\mathscr{H}}_{n,p} \longrightarrow \check{\mathscr{H}}_{n-1,p}, \qquad T \longmapsto T^*. \tag{9.4}$$

It is known that the above map (9.4) is a surjective map ([81] Theorem 2).

Let $\Psi^0_{n,r} : J_{\rho,\mathcal{M}}(\Gamma_n) \longrightarrow J_{\rho_0^{(r)},\mathcal{M}}(\Gamma_r)$ be the *modified Siegel–Jacobi operator* defined by

$$\left(\Psi^0_{n,r} f\right)(\Omega, Z) := \lim_{t\longrightarrow\infty} f\left(\begin{pmatrix} itI_{n-r} & 0 \\ 0 & \Omega \end{pmatrix}, (0, Z)\right), \quad (\Omega, Z) \in \mathbb{H}_{r,m},$$

where $\rho_0^{(r)} : GL(r,\mathbb{C}) \longrightarrow GL(V_\rho)$ is a finite-dimensional representation of $GL(r,\mathbb{C})$ defined by

$$\rho_0^{(r)}(A) = \begin{pmatrix} I_{n-r} & 0 \\ 0 & A \end{pmatrix}, \quad A \in GL(r,\mathbb{C}).$$

In [62], Yang proved that the action of the Hecke operators is compatible with that of the Siegel–Jacobi operator:

Theorem 9.4. *Suppose we have*

(a) *a rational finite-dimensional representation*

$$\rho : GL(n,\mathbb{C}) \longrightarrow GL(V_\rho),$$

(b) *a rational finite-dimensional representation*

$$\rho_0 : GL(n-1,\mathbb{C}) \longrightarrow GL(V_{\rho_0}),$$

(c) *a linear map* $R : V_\rho \longrightarrow V_{\rho_0}$,

satisfying the following properties (1) *and* (2)*:*

(1) $R \circ \rho \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} = \rho_0(A) \circ R$ *for all* $A \in GL(n-1,\mathbb{C})$,

(2) $R \circ \rho \begin{pmatrix} a & 0 \\ 0 & I_{n-1} \end{pmatrix} = a^k R$ *for some* $k \in \mathbb{Z}$.

Then for any $f \in J_{\rho,\mathcal{M}}(\Gamma_n)$ *and* $T \in \check{\mathscr{H}}_{n,p}$, *we have*

$$\left(R \circ \Psi^0_{n,n-1}\right)(f|T) = R(\Psi^0_{n,n-1} f)|T^*.$$

Proof. See Theorem 4.2 in [62]. □

Remark 9.1. Freitag [14] introduced the concept of stable modular forms using the Siegel operator and developed the theory of stable modular forms. We can define the concept of stable Jacobi forms using the Siegel–Jacobi operator and develop the theory of stable Jacobi forms.

10. Construction of vector-valued modular forms from Jacobi forms

Let n and m be two positive integers and let $\mathcal{P}_{m,n} := \mathbb{C}[z_{11}, \ldots, z_{mn}]$ be the ring of complex-valued polynomials on $\mathbb{C}^{(m,n)}$. For any homogeneous polynomial $P \in \mathcal{P}_{m,n}$, we put

$$P(\partial_Z) := P\left(\frac{\partial}{\partial z_{11}}, \ldots, \frac{\partial}{\partial z_{11}}\right).$$

Let S be a positive definite symmetric rational matrix of degree m. Let $T := (t_{pq})$ be the inverse of S. For each i, j with $1 \leq i, j \leq n$, we denote by $\Delta_{i,j}$ the following differential operator

$$\Delta_{i,j} := \sum_{p,q=1}^{m} t_{pq} \frac{\partial^2}{\partial z_{pi} \partial z_{qj}}, \qquad 1 \leq i, j \leq n.$$

A polynomial P on $\mathbb{C}^{(m,n)}$ is said to be *harmonic* with respect to S if

$$\sum_{i=1}^{n} \Delta_{i,i} P = 0.$$

A polynomial P on $\mathbb{C}^{(m,n)}$ is said to be *pluriharmonic* with respect to S if

$$\Delta_{i,j} P = 0, \qquad 1 \leq i, j \leq n.$$

If there is no confusion, we just write harmonic or pluriharmonic instead of harmonic or pluriharmonic with respect to S. Obviously a pluriharmonic polynomial is harmonic. We denote by $\mathscr{H}_{m,n}$ the space of all pluriharmonic polynomials on $\mathbb{C}^{(m,n)}$. The ring $\mathcal{P}_{m,n}$ has a symmetric nondegenerate bilinear form $\langle P, Q \rangle := \big(P(\partial_Z)Q\big)(0)$ for $P, Q \in \mathcal{P}_{m,n}$. It is easy to check that $\langle\ ,\ \rangle$ satisfies

$$\langle P, QR \rangle = \langle Q(\partial_Z)P, R \rangle, \qquad P, Q, R \in \mathcal{P}_{m,n}.$$

Lemma 10.1. *$\mathscr{H}_{m,n}$ is invariant under the action of $GL(n,\mathbb{C}) \times O(S)$ given by*

$$\big((A,B), P(Z)\big) \longmapsto P({}^tBZA), \qquad A \in GL(n,\mathbb{C}),\ B \in O(S),\ P \in \mathscr{H}_{m,n}. \quad (10.1)$$

Here $O(S) := \big\{ B \in GL(m,\mathbb{C}) \mid {}^tBSB = S \big\}$ denotes the orthogonal group of the quadratic form S.

Proof. See Corollary 9.11 in [45]. □

Remark 10.1. In [28], Kashiwara and Vergne investigated an irreducible decomposition of the space of complex pluriharmonic polynomials defined on $\mathbb{C}^{(m,n)}$ under the action (10.1). They showed that each irreducible component $\tau \otimes \lambda$ occurring in the decomposition of $\mathscr{H}_{m,n}$ under the action (10.1) has multiplicity one and the irreducible representation τ of $GL(n,\mathbb{C})$ is determined uniquely by the irreducible representation of $O(S)$.

Throughout this section we fix a rational representation ρ of $GL(n,\mathbb{C})$ on a finite-dimensional complex vector space V_ρ and a positive definite symmetric, half-integral matrix $\mathcal{M}$ of degree m once and for all.

Definition 10.1. A holomorphic function $f : \mathbb{H}_n \longrightarrow V_\rho$ is called a *modular form of type ρ* on Γ_n if

$$f(M \cdot \Omega) = f\big((A\Omega + B)(C\Omega + D)^{-1}\big) = \rho(C\Omega + D) f(\Omega), \quad \Omega \in \mathbb{H}_n$$

for all $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$. If $n = 1$, the additional cuspidal condition will be added. We denote by $[\Gamma_n, \rho]$ the vector space of all modular forms of type ρ on Γ_n.

Let $\mathscr{H}_{m,n;\mathcal{M}}$ be the vector space of all pluriharmonic polynomials on $\mathbb{C}^{(m,n)}$ with respect to $S := (2\mathcal{M})^{-1}$. According to Lemma 10.1, there exists an irreducible subspace $V_\tau (\neq 0)$ invariant under the action of $GL(n,\mathbb{C})$ given by (10.1). We denote this representation by τ. Then we have

$$\big(\tau(A)P\big)(Z) = P(ZA), \qquad A \in GL(n,\mathbb{C}),\ P \in V_\tau,\ Z \in \mathbb{C}^{(m,n)}.$$

The action $\widehat{\tau}$ of $GL(n,\mathbb{C})$ on V_τ^* is defined by

$$\big(\widehat{\tau}(A)^{-1}\zeta\big)(P) := \zeta\big(\tau({}^tA^{-1})P\big),$$

where $A \in GL(n,\mathbb{C}),\ \zeta \in V_\tau^*$ and $P \in V_\tau$.

Definition 10.2. Let $f \in J_{\rho,\mathcal{M}}(\Gamma_n)$ be a Jacobi form of index $\mathcal{M}$ with respect to ρ on Γ_n. Let $P \in V_\tau$ be a homogeneous pluriharmonic polynomial. We put

$$f_P(\Omega) := P(\partial_Z) f(\Omega, Z)|_{Z=0}, \qquad \Omega \in \mathbb{H}_n,\ Z \in \mathbb{C}^{(m,n)}.$$

Now we define the mapping

$$f_\tau : \mathbb{H}_n \longrightarrow V_\tau^* \otimes V_\rho$$

by

$$\big(f_\tau(\Omega)\big)(P) := f_P(\Omega), \qquad \Omega \in \mathbb{H}_n,\ P \in V_\tau. \tag{10.2}$$

Yang proved the following theorem in [66].

Theorem 10.1. *Let τ and $\widehat{\tau}$ be as before. Let $f \in J_{\rho,\mathcal{M}}(\Gamma_n)$ be a Jacobi form of index $\mathcal{M}$ with respect to ρ on Γ_n. Then $f_\tau(\Omega)$ is a modular form of type $\widehat{\tau} \otimes \rho$, i.e., $f_\tau \in [\Gamma_n, \widehat{\tau} \otimes \rho]$.*

Proof. See Main Theorem in [66]. □

We obtain an interesting and important identity by applying Theorem 10.1 to the Eisenstein series. Let $\mathcal{M}$ be a half-integral positive symmetric matrix of degree m. We set

$$\Gamma_{n;[0]} := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n \;\Big|\; C = 0 \right\}.$$

Let $\mathscr{R}$ be a complete system of representatives of the cosets $\Gamma_{n;[0]}\backslash\Gamma_n$ and Λ be a complete system of representatives of the cosets $\mathbb{Z}^{(m,n)}/\big(\mathrm{Ker}(\mathcal{M}) \cap \mathbb{Z}^{(m,n)}\big)$, where $\mathrm{Ker}(\mathcal{M}) := \big\{ \lambda \in \mathbb{R}^{(m,n)} \mid \mathcal{M}\cdot\lambda = 0 \big\}$. Let $k \in \mathbb{Z}^+$ be a positive integer. In [82], Ziegler defined the Eisenstein series $E^{(n)}_{k,\mathcal{M}}(\Omega, Z)$ of Siegel type by

$$\begin{aligned} E^{(n)}_{k,\mathcal{M}}(\Omega, Z) := &\sum_{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathscr{R}} \det(C\Omega + D)^{-k} \cdot e^{2\pi i\,\sigma(\mathcal{M} Z (C\Omega+D)^{-1} C\, {}^tZ)} \\ &\cdot \sum_{\lambda \in \Lambda} e^{2\pi i\,\sigma\left(\mathcal{M}((A\Omega+B)(C\Omega+D)^{-1}\, {}^t\lambda + 2\,\lambda\, {}^t(C\Omega+D)^{-1}\, {}^tZ)\right)}, \end{aligned}$$

where $(\Omega, Z) \in \mathbb{H}_{n,m}$. Now we assume that $k > n + m + 1$ and k is even. Then according to [82], Theorem 2.1, $E^{(n)}_{k,\mathcal{M}}(\Omega, Z)$ is a nonvanishing Jacobi form in $J_{k,\mathcal{M}}(\Gamma_n)$. By Theorem 10.1, $\big(E^{(n)}_{k,\mathcal{M}}\big)_\tau$ is a $\mathrm{Hom}(V_\tau,\mathbb{C})$-valued modular form of type $\widehat{\tau}\otimes\det^k$. We define the automorphic factor $j : Sp(n,\mathbb{R})\times\mathbb{H}_n \longrightarrow GL(n,\mathbb{C})$ by

$$j(g,\Omega) := C\Omega + D, \qquad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R}),\ \Omega \in \mathbb{H}_n.$$

Then according to the relation occurring in the process of the proof of Theorem 10.1, for any homogeneous pluriharmonic polynomial P with respect to $(2\,\mathcal{M})^{-1}$, we obtain the following identity

$$\begin{aligned}\det j(M,\Omega)^k \sum_{\gamma\in\mathscr{R}}\sum_{\lambda\in\Lambda} \det j(\gamma,\Omega)^{-k}\cdot P\big(4\pi i\,\mathcal{M}\lambda\, {}^t\!j(\gamma,\Omega)^{-1}\big)\cdot e^{2\pi i\,\sigma\left(\mathcal{M}(\gamma\cdot\Omega)\,{}^t\lambda\right)} \qquad (10.3)\\ = \sum_{\gamma\in\mathscr{R}}\sum_{\lambda\in\Lambda} \det j(\gamma, M\cdot\Omega)^{-k}\cdot P\big(4\pi i\,\mathcal{M}\lambda\, {}^t\!j(\gamma M,\Omega)^{-1}\big)\cdot e^{2\pi i\,\sigma\left(\mathcal{M}((\gamma M)\cdot\Omega)\,{}^t\lambda\right)}\end{aligned}$$

for all $M\in\Gamma_n$ and $\Omega\in\mathbb{H}_n$.

For any homogeneous pluriharmonic polynomial P with respect to $(2\,\mathcal{M})^{-1}$, we define the function $G_P:\Gamma_n\times\mathbb{H}_n\longrightarrow\mathbb{C}$ by

$$G_P(M,\Omega) := \sum_{\gamma\in\mathscr{R}}\sum_{\lambda\in\Lambda} \det j(\gamma M,\Omega)^{-k} P\big(4\pi i\mathcal{M}\lambda\,{}^t\!j(\gamma M,\Omega)^{-1}\big) e^{2\pi i\sigma\left(\mathcal{M}((\gamma M)\cdot\Omega)\,{}^t\lambda\right)}, \qquad (10.4)$$

where $M\in\Gamma_n$ and $\Omega\in\mathbb{H}_n$. Then according to Formula (10.3), we obtain the following relation

$$G_P(M,\Omega) = G_P(I_{2n},\Omega) \qquad \text{for all } M\in\Gamma_n \text{ and } \Omega\in\mathbb{H}_n. \qquad (10.5)$$

If $P=c$ is a constant, we see from (10.3) and (10.5) that $G_c := G_P$ satisfies the following relation

$$G_c(M, N\cdot\Omega) = G_c(I_{2n}, N\cdot\Omega) = \det j(N,\Omega)^k\, G_c(M,\Omega) \qquad (10.6)$$

for all $M,N\in\Gamma_n$ and $\Omega\in\mathbb{H}_n$. Therefore for any $M\in\Gamma_n$, the function $G_c(M,\cdot):\mathbb{H}_n\longrightarrow\mathbb{C}$ is a Siegel modular form of weight k.

11. Maass–Jacobi forms

Using G^J-invariant differential operators on the Siegel–Jacobi space, we introduce a notion of Maass–Jacobi forms.

Definition 11.1. Let

$$\Gamma_{n,m} := Sp(n,\mathbb{Z}) \ltimes H^{(n,m)}_{\mathbb{Z}}$$

be the discrete subgroup of G^J, where

$$H^{(n,m)}_{\mathbb{Z}} = \left\{ (\lambda,\mu;\kappa)\in H^{(n,m)}_{\mathbb{R}} \mid \lambda,\mu,\kappa \text{ are integral} \right\}.$$

A smooth function $f:\mathbb{H}_n\times\mathbb{C}^{(m,n)}\longrightarrow\mathbb{C}$ is called a Maass–Jacobi form on $\mathbb{H}_n\times\mathbb{C}^{(m,n)}$ if f satisfies the following conditions (MJ1)–(MJ3):

(MJ1) f is invariant under $\Gamma_{n,m}$.

(MJ2) f is an eigenfunction of the Laplacian $\Delta_{n,m;A,B}$ (cf. Formula (2.4)).

(MJ3) f has a polynomial growth, that is, there exist a constant $C > 0$ and a positive integer N such that

$$|f(X+i\,Y,Z)| \le C\,|p(Y)|^N \quad \text{as } \det Y \longrightarrow \infty,$$

where $p(Y)$ is a polynomial in $Y=(y_{ij})$.

Remark 11.1. We also may define the notion of Maass–Jacobi forms as follows. Let $\mathbb{D}_*$ be a commutative subalgebra of $\mathbb{D}(\mathbb{H}_n\times\mathbb{C}^{(m,n)})$ containing the Laplacian $\Delta_{n,m;A,B}$. We say that a smooth function $f:\mathbb{H}_n\times\mathbb{C}^{(m,n)}\longrightarrow\mathbb{C}$ is a Maass–Jacobi form with respect to $\mathbb{D}_*$ if f satisfies the conditions (MJ1), $(\text{MJ2})_*$ and (MJ3): the condition $(\text{MJ2})_*$ is given by

$(\text{MJ2})_*$ f is an eigenfunction of any invariant differential operator in $\mathbb{D}_*$.

Remark 11.2. Erik Balslev [2] developed the spectral theory of $\Delta_{1,1;1,1}$ on $\mathbb{H}_{1,1}$ to prove that the set of all eigenvalues of $\Delta_{1,1;1,1}$ satisfies the Weyl law.

It is natural to propose the following problems.

Problem A: Find all the eigenfunctions of $\Delta_{n,m;A,B}$.

Problem B: Construct Maass–Jacobi forms.

If we find a *nice* eigenfunction ϕ of the Laplacian $\Delta_{n,m;A,B}$, we can construct a Maass–Jacobi form f_ϕ on $\mathbb{H}_n\times\mathbb{C}^{(m,n)}$ in the usual way defined by

$$f_\phi(\Omega,Z) := \sum_{\gamma\in\Gamma^\infty_{n,m}\backslash\Gamma_{n,m}} \phi\big(\gamma\cdot(\Omega,Z)\big),$$

where

$$\Gamma^\infty_{n,m} = \left\{ \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, (\lambda,\mu;\kappa) \right) \in \Gamma_{n,m} \,\middle|\, C=0 \right\}$$

is a subgroup of $\Gamma_{n,m}$.

We consider the simple case when $n=m=1$ and $A=B=1$. A metric $ds^2_{1,1;1,1}$ on $\mathbb{H}_{1,1}$ given by

$$ds^2_{1,1;1,1} = \frac{y+v^2}{y^3}\,(\,dx^2+dy^2\,) + \frac{1}{y}\,(\,du^2+dv^2\,) - \frac{2v}{y^2}\,(\,dx\,du + dy\,dv\,)$$

is a G^J-invariant Kähler metric on $\mathbb{H}_{1,1}$. Its Laplacian $\Delta_{1,1;1,1}$ is given by

$$\Delta_{1,1;1,1} = y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) + (y+v^2)\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right) + 2yv\left(\frac{\partial^2}{\partial x\partial u}+\frac{\partial^2}{\partial y\partial v}\right).$$

We provide some examples of eigenfunctions of $\Delta_{1,1;1,1}$.

(a) $h(x,y) = y^{\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|a|y)\,e^{2\pi i a x}$ $(\,s\in\mathbb{C},\ a\neq 0\,)$ with eigenvalue $s(s-1)$. Here

$$K_s(z) := \frac{1}{2}\int_0^\infty \exp\left\{-\frac{z}{2}(t+t^{-1})\right\}\,t^{s-1}\,dt,$$

where $\operatorname{Re} z>0$.

(b) $y^s,\ y^s x,\ y^s u$ $(s\in\mathbb{C})$ with eigenvalue $s(s-1)$.

(c) $y^s v,\ y^s uv,\ y^s xv$ with eigenvalue $s(s+1)$.

(d) x, y, u, v, xv, uv with eigenvalue 0.
(e) All Maass wave forms.

Let ρ be a rational representation of $GL(n,\mathbb{C})$ on a finite-dimensional complex vector space V_ρ. Let $\mathcal{M} \in \mathbb{R}^{(m,m)}$ be a symmetric half-integral semi-positive definite matrix of degree m. Let $C^\infty(\mathbb{H}_n \times \mathbb{C}^{(m,n)}, V_\rho)$ be the algebra of all C^∞ functions on $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$ with values in V_ρ. We define the $|_{\rho,\mathcal{M}}$-slash action of G^J on $C^\infty(\mathbb{H}_n \times \mathbb{C}^{(m,n)}, V_\rho)$ as follows: If $f \in C^\infty(\mathbb{H}_n \times \mathbb{C}^{(m,n)}, V_\rho)$,

$$\begin{aligned} f|_{\rho,\mathcal{M}}[(M,(\lambda,\mu;\kappa))](\Omega,Z) \\ := e^{-2\pi i\,\sigma(\mathcal{M}[Z+\lambda\Omega+\mu](C\Omega+D)^{-1}C)} \cdot e^{2\pi i\,\sigma(\mathcal{M}(\lambda\Omega\,{}^t\lambda\,+\,2\lambda\,{}^tZ\,+\,\kappa\,+\,\mu\,{}^t\lambda))} \\ \times\ \rho(C\Omega+D)^{-1} f(M\cdot\Omega,(Z+\lambda\Omega+\mu)(C\Omega+D)^{-1}), \end{aligned}$$

where $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp(n,\mathbb{R})$ and $(\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)}$. We recall the Siegel's notation $\alpha[\beta] = {}^t\beta\alpha\beta$ for suitable matrices α and β. We define $\mathbb{D}_{\rho,\mathcal{M}}$ to be the algebra of all differential operators D on $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$ satisfying the following condition

$$(Df)|_{\rho,\mathcal{M}}[g] = D(f|_{\rho,\mathcal{M}}[g])$$

for all $f \in C^\infty(\mathbb{H}_n \times \mathbb{C}^{(m,n)}, V_\rho)$ and for all $g \in G^J$. We denote by $\mathcal{Z}_{\rho,\mathcal{M}}$ the center of $\mathbb{D}_{\rho,\mathcal{M}}$.

We define another notion of Maass–Jacobi forms as follows.

Definition 11.2. A vector-valued smooth function $\phi : \mathbb{H}_n \times \mathbb{C}^{(m,n)} \longrightarrow V_\rho$ is called a Maass–Jacobi form on $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$ of type ρ and index $\mathcal{M}$ if it satisfies the following conditions $(\mathrm{MJ1})_{\rho,\mathcal{M}}$, $(\mathrm{MJ2})_{\rho,\mathcal{M}}$ and $(\mathrm{MJ3})_{\rho,\mathcal{M}}$:

$(\mathrm{MJ1})_{\rho,\mathcal{M}}$ $\phi|_{\rho,\mathcal{M}}[\gamma] = \phi$ for all $\gamma \in \Gamma_{n,m}$.

$(\mathrm{MJ2})_{\rho,\mathcal{M}}$ f is an eigenfunction of all differential operators in the center $\mathcal{Z}_{\rho,\mathcal{M}}$ of $\mathbb{D}_{\rho,\mathcal{M}}$.

$(\mathrm{MJ3})_{\rho,\mathcal{M}}$ f has a growth condition

$$\phi(\Omega,Z) = O\Big(e^{a\det Y} \cdot e^{2\pi \mathrm{tr}(\mathcal{M}[V]Y^{-1})}\Big)$$

as $\det Y \longrightarrow \infty$ for some $a > 0$.

Remark 11.3. In the sense of Definition 11.2, Pitale [50] studied Maass–Jacobi forms on the Siegel–Jacobi space $\mathbb{H}_{1,1}$. We refer to [74, 75] for more details on Maass–Jacobi forms.

12. The Schrödinger–Weil representation

Throughout this section we assume that $\mathcal{M}$ is a positive definite symmetric real $m \times m$ matrix. We consider the Schrödinger representation $\mathscr{W}_\mathcal{M}$ of the Heisenberg group $H_{\mathbb{R}}^{(n,m)}$ with the central character $\mathscr{W}_\mathcal{M}((0,0;\kappa)) = \chi_\mathcal{M}((0,0;\kappa)) = e^{\pi i\,\sigma(\mathcal{M}\kappa)}$, $\kappa \in S(m,\mathbb{R})$. Then $\mathscr{W}_\mathcal{M}$ is expresses explicitly as follows:

$$[\mathscr{W}_\mathcal{M}(h_0)f]\,(\lambda) = e^{\pi i \sigma\{\mathcal{M}(\kappa_0+\mu_0\,{}^t\lambda_0+2\lambda\,{}^t\mu_0)\}}\, f(\lambda+\lambda_0), \tag{12.1}$$

where $h_0 = (\lambda_0, \mu_0; \kappa_0) \in H_{\mathbb{R}}^{(n,m)}$ and $\lambda \in \mathbb{R}^{(m,n)}$. For the construction of $\mathscr{W}_{\mathcal{M}}$ we refer to [78]. We note that the symplectic group $Sp(n, \mathbb{R})$ acts on $H_{\mathbb{R}}^{(n,m)}$ by conjugation inside G^J. For a fixed element $g \in Sp(n, \mathbb{R})$, the irreducible unitary representation $\mathscr{W}_{\mathcal{M}}^g$ of $H_{\mathbb{R}}^{(n,m)}$ defined by

$$\mathscr{W}_{\mathcal{M}}^g(h) = \mathscr{W}_{\mathcal{M}}(ghg^{-1}), \quad h \in H_{\mathbb{R}}^{(n,m)} \tag{12.2}$$

has the property that

$$\mathscr{W}_{\mathcal{M}}^g((0,0;\kappa)) = \mathscr{W}_{\mathcal{M}}((0,0;\kappa)) = e^{\pi i\, \sigma(\mathcal{M}\kappa)}\, \mathrm{Id}_{H(\chi_{\mathcal{M}})}, \quad \kappa \in S(m, \mathbb{R}).$$

Here $\mathrm{Id}_{H(\chi_{\mathcal{M}})}$ denotes the identity operator on the Hilbert space $H(\chi_{\mathcal{M}})$. According to the Stone–von Neumann theorem, there exists a unitary operator $R_{\mathcal{M}}(g)$ on $H(\chi_{\mathcal{M}})$ with $R_{\mathcal{M}}(I_{2n}) = \mathrm{Id}_{H(\chi_{\mathcal{M}})}$ such that

$$R_{\mathcal{M}}(g)\mathscr{W}_{\mathcal{M}}(h) = \mathscr{W}_{\mathcal{M}}^g(h)R_{\mathcal{M}}(g) \qquad \text{for all } h \in H_{\mathbb{R}}^{(n,m)}. \tag{12.3}$$

We observe that $R_{\mathcal{M}}(g)$ is determined uniquely up to a scalar of modulus one.

From now on, for brevity, we put $G = Sp(n, \mathbb{R})$. According to Schur's lemma, we have a map $c_{\mathcal{M}} : G \times G \longrightarrow T$ satisfying the relation

$$R_{\mathcal{M}}(g_1 g_2) = c_{\mathcal{M}}(g_1, g_2) R_{\mathcal{M}}(g_1) R_{\mathcal{M}}(g_2) \quad \text{for all } \ g_1, g_2 \in G. \tag{12.4}$$

We recall that T denotes the multiplicative group of complex numbers of modulus one. Therefore $R_{\mathcal{M}}$ is a projective representation of G on $H(\chi_{\mathcal{M}})$ and $c_{\mathcal{M}}$ defines the cocycle class in $H^2(G, T)$. The cocycle $c_{\mathcal{M}}$ yields the central extension $G_{\mathcal{M}}$ of G by T. The group $G_{\mathcal{M}}$ is a set $G \times T$ equipped with the following multiplication

$$(g_1, t_1) \cdot (g_2, t_2) = \big(g_1 g_2, t_1 t_2\, c_{\mathcal{M}}(g_1, g_2)^{-1}\big), \quad g_1, g_2 \in G,\ t_1, t_2 \in T. \tag{12.5}$$

We see immediately that the map $\widetilde{R}_{\mathcal{M}} : G_{\mathcal{M}} \longrightarrow GL(H(\chi_{\mathcal{M}}))$ defined by

$$\widetilde{R}_{\mathcal{M}}(g, t) = t\, R_{\mathcal{M}}(g) \quad \text{for all } (g, t) \in G_{\mathcal{M}} \tag{12.6}$$

is a *true* representation of $G_{\mathcal{M}}$. As in Section 1.7 in [35], we can define the map $s_{\mathcal{M}} : G \longrightarrow T$ satisfying the relation

$$c_{\mathcal{M}}(g_1, g_2)^2 = s_{\mathcal{M}}(g_1)^{-1} s_{\mathcal{M}}(g_2)^{-1} s_{\mathcal{M}}(g_1 g_2) \quad \text{for all } g_1, g_2 \in G.$$

Thus we see that

$$G_{2,\mathcal{M}} = \big\{\, (g, t) \in G_{\mathcal{M}} \mid t^2 = s_{\mathcal{M}}(g)^{-1} \,\big\} \tag{12.7}$$

is the metaplectic group associated with $\mathcal{M}$ that is a two-fold covering group of G. The restriction $R_{2,\mathcal{M}}$ of $\widetilde{R}_{\mathcal{M}}$ to $G_{2,\mathcal{M}}$ is the Weil representation of G associated with $\mathcal{M}$.

If we identify $h = (\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(n,m)}$ (resp. $g \in Sp(n, \mathbb{R})$) with $(I_{2n}, (\lambda, \mu; \kappa)) \in G^J$ (resp. $(g, (0,0;0)) \in G^J$), every element $\tilde{g}$ of G^J can be written as $\tilde{g} = hg$ with $h \in H_{\mathbb{R}}^{(n,m)}$ and $g \in Sp(n, \mathbb{R})$. In fact,

$$(g, (\lambda, \mu; \kappa)) = (I_{2n}, ((\lambda, \mu)g^{-1}; \kappa))\, (g, (0,0;0)) = ((\lambda, \mu)g^{-1}; \kappa) \cdot g.$$

Therefore we define the *projective* representation $\pi_{\mathcal{M}}$ of the Jacobi group G^J with cocycle $c_{\mathcal{M}}(g_1, g_2)$ by

$$\pi_{\mathcal{M}}(hg) = \mathscr{W}_{\mathcal{M}}(h)\, R_{\mathcal{M}}(g), \quad h \in H_{\mathbb{R}}^{(n,m)},\ g \in G. \tag{12.8}$$

Indeed, since $H_{\mathbb{R}}^{(n,m)}$ is a normal subgroup of G^J, for any $h_1, h_2 \in H_{\mathbb{R}}^{(n,m)}$ and $g_1, g_2 \in G$,

$$\begin{aligned}\pi_{\mathcal{M}}(h_1g_1h_2g_2) &= \pi_{\mathcal{M}}(h_1g_1h_2g_1^{-1}g_1g_2) = \mathscr{W}_{\mathcal{M}}\big(h_1(g_1h_2g_1^{-1})\big)R_{\mathcal{M}}(g_1g_2)\\ &= c_{\mathcal{M}}(g_1,g_2)\mathscr{W}_{\mathcal{M}}(h_1)\mathscr{W}_{\mathcal{M}}^{g_1}(h_2)R_{\mathcal{M}}(g_1)R_{\mathcal{M}}(g_2)\\ &= c_{\mathcal{M}}(g_1,g_2)\mathscr{W}_{\mathcal{M}}(h_1)R_{\mathcal{M}}(g_1)\mathscr{W}_{\mathcal{M}}(h_2)R_{\mathcal{M}}(g_2)\\ &= c_{\mathcal{M}}(g_1,g_2)\pi_{\mathcal{M}}(h_1g_1)\pi_{\mathcal{M}}(h_2g_2).\end{aligned}$$

We let

$$G_{\mathcal{M}}^J = G_{\mathcal{M}} \ltimes H_{\mathbb{R}}^{(n,m)}$$

be the semidirect product of $G_{\mathcal{M}}$ and $H_{\mathbb{R}}^{(n,m)}$ with the multiplication law

$$\begin{aligned}&\big((g_1,t_1),(\lambda_1,\mu_1;\kappa_1)\big)\cdot\big((g_2,t_2),(\lambda_2,\mu_2\,;\kappa_2)\big)\\ &\quad = \big((g_1,t_1)(g_2,t_2),(\tilde\lambda+\lambda_2,\tilde\mu+\mu_2\,;\kappa_1+\kappa_2+\tilde\lambda\,{}^t\mu_2-\tilde\mu\,{}^t\lambda_2)\big),\end{aligned}$$

where ⟸ disp

$$(g_1,t_1),(g_2,t_2)\in G_{\mathcal{M}}, \quad (\lambda_1,\mu_1;\kappa_1),(\lambda_2,\mu_2;\kappa_2)\in H_{\mathbb{R}}^{(n,m)} \quad \text{and} \quad (\tilde\lambda,\tilde\mu)=(\lambda,\mu)g_2.$$

If we identify $h = (\lambda,\mu\,;\kappa) \in H_{\mathbb{R}}^{(n,m)}$ (resp. $(g,t) \in G_{\mathcal{M}}$) with $((I_{2n},1),(\lambda,\mu\,;\kappa)) \in G_{\mathcal{M}}^J$ (resp. $((g,t),(0,0;0)) \in G_{\mathcal{M}}^J$), we see easily that every element $\big((g,t),(\lambda,\mu;\kappa)\big)$ of $G_{\mathcal{M}}^J$ can be expressed as

$$\big((g,t),(\lambda,\mu\,;\kappa)\big) = \big((I_{2n},1),((\lambda,\mu)g^{-1};\kappa)\big)\big((g,t),(0,0;0)\big) = ((\lambda,\mu)g^{-1};\kappa)(g,t).$$

Now we can define the *true* representation $\widetilde\omega_{\mathcal{M}}$ of $G_{\mathcal{M}}^J$ by

$$\widetilde\omega_{\mathcal{M}}(h\cdot(g,t)) = t\,\pi_{\mathcal{M}}(hg) = t\,\mathscr{W}_{\mathcal{M}}(h)\,R_{\mathcal{M}}(g), \quad h\in H_{\mathbb{R}}^{(n,m)},\ (g,t)\in G_{\mathcal{M}}. \tag{12.9}$$

Indeed, since $H_{\mathbb{R}}^{(n,m)}$ is a normal subgroup of $G_{\mathcal{M}}^J$,

$$\begin{aligned}&\widetilde\omega_{\mathcal{M}}\big(h_1(g_1,t_1)h_2(g_2,t_2)\big) = \widetilde\omega_{\mathcal{M}}\big(h_1(g_1,t_1)h_2(g_1,t_1)^{-1}(g_1,t_1)(g_2,t_2)\big)\\ &\quad= \widetilde\omega_{\mathcal{M}}\big(h_1(g_1,t_1)h_2(g_1,t_1)^{-1}\big(g_1g_2,t_1t_2\,c_{\mathcal{M}}(g_1,g_2)^{-1}\big)\big)\\ &\quad= t_1t_2\,c_{\mathcal{M}}(g_1,g_2)^{-1}\,\mathscr{W}_{\mathcal{M}}\big(h_1(g_1,t_1)h_2(g_1,t_1)^{-1}\big)R_{\mathcal{M}}(g_1g_2)\\ &\quad= t_1t_2\,\mathscr{W}_{\mathcal{M}}(h_1)\mathscr{W}_{\mathcal{M}}\big((g_1,t_1)h_2(g_1,t_1)^{-1}\big)\,R_{\mathcal{M}}(g_1)R_{\mathcal{M}}(g_2)\\ &\quad= t_1t_2\,\mathscr{W}_{\mathcal{M}}(h_1)\mathscr{W}_{\mathcal{M}}\big(g_1h_2g_1^{-1}\big)\,R_{\mathcal{M}}(g_1)R_{\mathcal{M}}(g_2)\\ &\quad= t_1t_2\,\mathscr{W}_{\mathcal{M}}(h_1)R_{\mathcal{M}}(g_1)\,\mathscr{W}_{\mathcal{M}}(h_2)R_{\mathcal{M}}(g_2)\\ &\quad= \{t_1\,\pi_{\mathcal{M}}(h_1g_1)\}\,\{t_2\,\pi_{\mathcal{M}}(h_2g_2)\}\\ &\quad= \widetilde\omega_{\mathcal{M}}\big(h_1(g_1,t_1)\big)\,\widetilde\omega_{\mathcal{M}}\big(h_2(g_2,t_2)\big).\end{aligned}$$

Here we used the fact that $(g_1,t_1)h_2(g_1,t_1)^{-1} = g_1h_2g_1^{-1}$.

We recall that the following matrices

$$t(b)=\begin{pmatrix} I_n & b\\ 0 & I_n\end{pmatrix}\quad\text{with any } b={}^tb\in\mathbb{R}^{(n,n)},$$
$$g(\alpha)=\begin{pmatrix} {}^t\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}\ \text{with any } \alpha\in GL(n,\mathbb{R}),$$
$$\sigma_n=\begin{pmatrix} 0 & -I_n\\ I_n & 0\end{pmatrix}$$

generate the symplectic group $G=Sp(n,\mathbb{R})$ (cf. [15, p. 326], [44, p. 210]). Therefore the following elements $h_t(\lambda,\mu\,;\kappa),\ t(b\,;t),\ g(\alpha\,;t)$ and $\sigma_{n\,;t}$ of $G_{\mathcal{M}}\ltimes H_{\mathbb{R}}^{(n,m)}$ defined by

$$\begin{aligned}
h_t(\lambda,\mu\,;\kappa)&=\big((I_{2n},t),(\lambda,\mu;\kappa)\big)\quad\text{with } t\in T,\ \lambda,\mu\in\mathbb{R}^{(m,n)}\text{ and }\kappa\in\mathbb{R}^{(m,m)},\\
t(b\,;t)&=\big((t(b),t),(0,0;0)\big)\quad\text{with any } b={}^tb\in\mathbb{R}^{(n,n)},\ t\in T,\\
g(\alpha\,;t)&=\big((g(\alpha),t),(0,0;0)\big)\ \text{with any } \alpha\in GL(n,\mathbb{R})\text{ and } t\in T,\\
\sigma_{n\,;t}&=((\sigma_n,t),(0,0;0))\quad\text{with } t\in T
\end{aligned}$$

generate the group $G_{\mathcal{M}}\ltimes H_{\mathbb{R}}^{(n,m)}$. We can show that the representation $\widetilde{\omega}_{\mathcal{M}}$ is realized on the representation $H(\chi_{\mathcal{M}})=L^2\big(\mathbb{R}^{(m,n)}\big)$ as follows: for each $f\in L^2\big(\mathbb{R}^{(m,n)}\big)$ and $x\in\mathbb{R}^{(m,n)}$, the actions of $\widetilde{\omega}_{\mathcal{M}}$ on the generators are given by

$$\big[\widetilde{\omega}_{\mathcal{M}}\big(h_t(\lambda,\mu\,;\kappa)\big)f\big]\,(x)=\,t\,e^{\pi i\,\sigma\{\mathcal{M}(\kappa+\mu\,{}^t\lambda+2\,x\,{}^t\mu)\}}\,f(x+\lambda),\tag{12.10}$$
$$\big[\widetilde{\omega}_{\mathcal{M}}\big(t(b\,;t)\big)f\big]\,(x)=t\,e^{\pi i\,\sigma(\mathcal{M}\,x\,b\,{}^tx)}f(x),\tag{12.11}$$
$$\big[\widetilde{\omega}_{\mathcal{M}}\big(g(\alpha\,;t)\big)f\big]\,(x)=t\,|\det\alpha|^{\frac{m}{2}}\,f(x\,{}^t\alpha),\tag{12.12}$$
$$\big[\widetilde{\omega}_{\mathcal{M}}\big(\sigma_{n\,;t}\big)f\big]\,(x)=t\,\big(\det\mathcal{M}\big)^{\frac{n}{2}}\int_{\mathbb{R}^{(m,n)}}f(y)\,e^{-2\,\pi i\,\sigma(\mathcal{M}\,y\,{}^tx)}\,dy.\tag{12.13}$$

Let

$$G_{2,\mathcal{M}}^J=G_{2,\mathcal{M}}\ltimes H_{\mathbb{R}}^{(n,m)}$$

be the semidirect product of $G_{2,\mathcal{M}}$ and $H_{\mathbb{R}}^{(n,m)}$. Then $G_{2,\mathcal{M}}^J$ is a subgroup of $G_{\mathcal{M}}^J$ which is a two-fold covering group of the Jacobi group G^J. The restriction $\omega_{\mathcal{M}}$ of $\widetilde{\omega}_{\mathcal{M}}$ to $G_{2,\mathcal{M}}^J$ is called the Schrödinger–Weil representation of G^J associated with $\mathcal{M}$.

We denote by $L^2_+\big(\mathbb{R}^{(m,n)}\big)$ $\big($resp. $L^2_-\big(\mathbb{R}^{(m,n)}\big)\big)$ the subspace of $L^2\big(\mathbb{R}^{(m,n)}\big)$ consisting of even (resp. odd) functions in $L^2\big(\mathbb{R}^{(m,n)}\big)$. According to Formulas (12.11)–(12.13), $R_{2,\mathcal{M}}$ is decomposed into representations of $R_{2,\mathcal{M}}^{\pm}$

$$R_{2,\mathcal{M}}=R_{2,\mathcal{M}}^{+}\oplus R_{2,\mathcal{M}}^{-},$$

where $R_{2,\mathcal{M}}^{+}$ and $R_{2,\mathcal{M}}^{-}$ are the even Weil representation and the odd Weil representation of G that are realized on $L^2_+\big(\mathbb{R}^{(m,n)}\big)$ and $L^2_-\big(\mathbb{R}^{(m,n)}\big)$ respectively. Obviously the center $\mathscr{Z}_{2,\mathcal{M}}^J$ of $G_{2,\mathcal{M}}^J$ is given by

$$\mathscr{Z}_{2,\mathcal{M}}^J=\big\{\big((I_{2n},1),(0,0;\kappa)\big)\in G_{2,\mathcal{M}}^J\,\big\}\cong S(m,\mathbb{R}).$$

We note that the restriction of $\omega_{\mathcal{M}}$ to $G_{2,\mathcal{M}}$ coincides with $R_{2,\mathcal{M}}$ and $\omega_{\mathcal{M}}(h) = \mathscr{W}_{\mathcal{M}}(h)$ for all $h \in H_{\mathbb{R}}^{(n,m)}$.

Remark 12.1. In the case $n = m = 1$, $\omega_{\mathcal{M}}$ is dealt in [10] and [36]. We refer to [16] and [28] for more details about the Weil representation $R_{2,\mathcal{M}}$.

Remark 12.2. The Schrödinger–Weil representation is applied to the theory of Maass–Jacobi forms [50].

Let $\mathcal{M}$ be a positive definite symmetric real matrix of degree m. We recall the Schrödinger representation $\mathscr{W}_{\mathcal{M}}$ of the Heisenberg group $H_{\mathbb{R}}^{(n,m)}$ associate with $\mathcal{M}$ given by Formula (12.1). We note that for an element $(\lambda, \mu; \kappa)$ of $H_{\mathbb{R}}^{(n,m)}$, we have the decomposition

$$(\lambda, \mu; \kappa) = (\lambda, 0; 0) \circ (0, \mu; 0) \circ (0, 0; \kappa - \lambda\, {}^t\mu).$$

We consider the embedding $\Phi_n : SL(2,\mathbb{R}) \longrightarrow Sp(n,\mathbb{R})$ defined by

$$\Phi_n\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) := \begin{pmatrix} aI_n & bI_n \\ cI_n & dI_n \end{pmatrix}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}). \tag{12.14}$$

For $x, y \in \mathbb{R}^{(m,n)}$, we put

$$(x,y)_{\mathcal{M}} := \sigma({}^t x \mathcal{M} y) \qquad \text{and} \qquad \|x\|_{\mathcal{M}} := \sqrt{(x,x)_{\mathcal{M}}}.$$

According to Formulas (12.11)–(12.13), for any $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}) \hookrightarrow Sp(n,\mathbb{R})$ and $f \in L^2\left(\mathbb{R}^{(m,n)}\right)$, we have the following explicit representation

$$[R_{\mathcal{M}}(M)f](x) = \begin{cases} |a|^{\frac{mn}{2}} e^{ab\|x\|^2_{\mathcal{M}}\pi i} f(ax) & \text{if } c = 0, \\ (\det \mathcal{M})^{\frac{n}{2}}\, |c|^{-\frac{mn}{2}} \int_{\mathbb{R}^{(m,n)}} e^{\frac{\alpha(M,x,y,\mathcal{M})}{c}\pi i} f(y)dy & \text{if } c \neq 0, \end{cases} \tag{12.15}$$

where

$$\alpha(M,x,y,\mathcal{M}) = a\,\|x\|^2_{\mathcal{M}} + d\,\|y\|^2_{\mathcal{M}} - 2(x,y)_{\mathcal{M}}.$$

Indeed, if $a = 0$ and $c \neq 0$, using the decomposition

$$M = \begin{pmatrix} 0 & -c^{-1} \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}$$

and if $a \neq 0$ and $c \neq 0$, using the decomposition

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c^{-1} \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} ac & ad \\ 0 & (ac)^{-1} \end{pmatrix},$$

we obtain Formula (12.15).

If

$$M_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \quad \text{and} \quad M_3 = \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} \in SL(2,\mathbb{R})$$

with $M_3 = M_1 M_2$, the corresponding cocycle is given by

$$c_{\mathcal{M}}(M_1, M_2) = e^{-i\,\pi\, mn\, \mathrm{sign}(c_1 c_2 c_3)/4}, \tag{12.16}$$

where

$$\operatorname{sign}(x) = \begin{cases} -1 & (x < 0) \\ 0 & (x = 0) \\ 1 & (x > 0). \end{cases}$$

In the special case when

$$M_1 = \begin{pmatrix} \cos\phi_1 & -\sin\phi_1 \\ \sin\phi_1 & \cos\phi_1 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} \cos\phi_2 & -\sin\phi_2 \\ \sin\phi_2 & \cos\phi_2 \end{pmatrix},$$

we find

$$c_{\mathcal{M}}(M_1, M_2) = e^{-i\,\pi\, mn\,(\sigma_{\phi_1} + \sigma_{\phi_2} - \sigma_{\phi_1+\phi_2})/4},$$

where

$$\sigma_\phi = \begin{cases} 2\nu & \text{if } \phi = \nu\pi \\ 2\nu + 1 & \text{if } \nu\pi < \phi < (\nu+1)\pi. \end{cases}$$

It is well known that every $M \in SL(2,\mathbb{R})$ admits the unique Iwasawa decomposition

$$M = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^{1/2} & 0 \\ 0 & v^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}, \tag{12.17}$$

where $\tau = u + iv \in \mathbb{H}_1$ and $\phi \in [0, 2\pi)$. This parametrization $M = (\tau, \phi)$ in $SL(2,\mathbb{R})$ leads to the natural action of $SL(2,\mathbb{R})$ on $\mathbb{H}_1 \times [0, 2\pi)$ defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau, \phi) := \left(\frac{a\tau + b}{c\tau + d},\ \phi + \arg(c\tau + d) \bmod 2\pi \right). \tag{12.18}$$

Lemma 12.1. *For two elements g_1 and g_2 in $SL(2,\mathbb{R})$, we let*

$$g_1 = \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1^{1/2} & 0 \\ 0 & v_1^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\phi_1 & -\sin\phi_1 \\ \sin\phi_1 & \cos\phi_1 \end{pmatrix}$$

and

$$g_2 = \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_2^{1/2} & 0 \\ 0 & v_2^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\phi_2 & -\sin\phi_2 \\ \sin\phi_2 & \cos\phi_2 \end{pmatrix}$$

be the Iwasawa decompositions of g_1 and g_2 respectively, where $u_1, u_2 \in \mathbb{R},\ v_1 > 0,\ v_2 > 0$ and $0 \le \phi_1, \phi_2 < 2\pi$. Let

$$g_3 = g_1 g_2 = \begin{pmatrix} 1 & u_3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_3^{1/2} & 0 \\ 0 & v_3^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\phi_3 & -\sin\phi_3 \\ \sin\phi_3 & \cos\phi_3 \end{pmatrix}$$

be the Iwasawa decomposition of $g_3 = g_1 g_2$. Then we have

$$u_3 = \frac{A}{(u_2 \sin\phi_1 + \cos\phi_1)^2 + (v_2 \sin\phi_1)^2},$$

$$v_3 = \frac{v_1 v_2}{(u_2 \sin\phi_1 + \cos\phi_1)^2 + (v_2 \sin\phi_1)^2}$$

and

$$\phi_3 = tan^{-1} \left[\frac{(v_2 \cos\phi_2 + u_2 \sin\phi_2)\tan\phi_1 + \sin\phi_2}{(-v_2 \sin\phi_2 + u_2 \cos\phi_2)\tan\phi_1 + \cos\phi_2} \right],$$

where

$$A = u_1(u_2 \sin\phi_1 + \cos\phi_1)^2 + (u_1 v_2 - v_1 u_2)\sin^2\phi_1 + v_1 u_2 \cos^2\phi_1 + v_1(u_2^2 + v_2^2 - 1)\sin\phi_1\cos\phi_1.$$

Proof. If $g \in SL(2,\mathbb{R})$ has the unique Iwasawa decomposition (12.17), then we get the following

$$\begin{aligned} a &= v^{1/2}\cos\phi + uv^{-1/2}\sin\phi, & d &= v^{-1/2}\cos\phi,\\ b &= -v^{1/2}\sin\phi + uv^{-1/2}\cos\phi, & u &= (ac+bd)\left(c^2+d^2\right)^{-1},\\ c &= v^{-1/2}\sin\phi, & v &= \left(c^2+d^2\right)^{-1}, \quad \tan\phi = \frac{c}{d}. \end{aligned}$$

We set

$$g_3 = g_1 g_2 = \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}.$$

Since

$$u_3 = (a_3c_3 + b_3d_3)\left(c_3^2+d_3^2\right)^{-1}, \quad v = \left(c_3^2+d_3^2\right)^{-1}, \quad \tan\phi_3 = \frac{c_3}{d_3},$$

by an easy computation, we obtain the desired results. □

Now we use the new coordinates $(\tau = u+iv, \phi)$ with $\tau \in \mathbb{H}_1$ and $\phi \in [0, 2\pi)$ in $SL(2,\mathbb{R})$. According to Formulas (12.11)–(12.13), the projective representation $R_{\mathcal{M}}$ of $SL(2,\mathbb{R}) \hookrightarrow Sp(n,\mathbb{R})$ reads in these coordinates $(\tau = u+iv, \phi)$ as follows:

$$[R_{\mathcal{M}}(\tau,\phi)f](x) = v^{\frac{mn}{4}}\, e^{u\|x\|^2_{\mathcal{M}}\pi\, i}\,[R_{\mathcal{M}}(i,\phi)f]\left(v^{1/2}x\right), \tag{12.19}$$

where $f \in L^2\left(\mathbb{R}^{(m,n)}\right),\ x \in \mathbb{R}^{(m,n)}$ and

$$[R_{\mathcal{M}}(i,\phi)f](x) = \begin{cases} f(x) & \text{if } \phi \equiv 0 \bmod 2\pi,\\ f(-x) & \text{if } \phi \equiv \pi \bmod 2\pi,\\ (\det\mathcal{M})^{\frac{n}{2}}\,|\sin\phi|^{-\frac{mn}{2}} \int_{\mathbb{R}^{(m,n)}} e^{B(x,y,\phi,\mathcal{M})\pi i} f(y)dy & \text{if } \phi \not\equiv 0 \bmod \pi. \end{cases} \tag{12.20}$$

Here

$$B(x,y,\phi,\mathcal{M}) = \frac{\left(\|x\|^2_{\mathcal{M}} + \|y\|^2_{\mathcal{M}}\right)\cos\phi - 2(x,y)_{\mathcal{M}}}{\sin\phi}.$$

Now we set

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We note that

$$\left[R_{\mathcal{M}}\left(i,\frac{\pi}{2}\right)f\right](x) = [R_{\mathcal{M}}(S)f](x) = (\det\mathcal{M})^{\frac{n}{2}} \int_{\mathbb{R}^{(m,n)}} f(y)\, e^{-2\,(x,\,y)_{\mathcal{M}}\,\pi\, i}\, dy \tag{12.21}$$

for $f \in L^2\left(\mathbb{R}^{(m,n)}\right)$.

Remark 12.3. For Schwartz functions $f \in \mathscr{S}\left(\mathbb{R}^{(m,n)}\right)$, we have

$$\lim_{\phi \longrightarrow 0\pm} |\sin\phi|^{-\frac{mn}{2}} \int_{\mathbb{R}^{(m,n)}} e^{B(x,y,\phi,\mathcal{M})\,\pi\, i} f(y)dy = e^{\pm i\,\pi\, mn/4} f(x) \neq f(x).$$

Therefore the projective representation $R_{\mathcal{M}}$ is not continuous at $\phi = \nu\pi\,(\nu \in \mathbb{Z})$ in general. If we set

$$\tilde{R}_{\mathcal{M}}(\tau,\phi) = e^{-i\,\pi\, mn\sigma_\phi/4} R_{\mathcal{M}}(\tau,\phi),$$

$\tilde{R}_{\mathcal{M}}$ corresponds to a unitary representation of the double cover of $SL(2,\mathbb{R})$ (cf. (3.5) and [35]). This means in particular that

$$\tilde{R}_{\mathcal{M}}(i,\phi)\tilde{R}_{\mathcal{M}}(i,\phi') = \tilde{R}_{\mathcal{M}}(i,\phi+\phi'),$$

where $\phi \in [0,4\pi)$ parametrises the double cover of $SO(2) \subset SL(2,\mathbb{R})$.

We observe that for any element $(g,(\lambda,\mu;\kappa)) \in G^J$ with $g \in Sp(n,\mathbb{R})$ and $(\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)}$, we have the following decomposition

$$(g,(\lambda,\mu;\kappa)) = (I_{2n},((\lambda,\mu)g^{-1};\kappa))\,(g,(0,0;0)) = ((\lambda,\mu)g^{-1};\kappa)\cdot g.$$

Thus $Sp(n,\mathbb{R})$ acts on $H_{\mathbb{R}}^{(n,m)}$ naturally by

$$g\cdot(\lambda,\mu;\kappa) = \left((\lambda,\mu)g^{-1};\kappa\right), \qquad g \in Sp(n,\mathbb{R}),\ (\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)}.$$

Definition 12.1. For any Schwartz function $f \in \mathscr{S}\left(\mathbb{R}^{(m,n)}\right)$, we define the function $\Theta_f^{[\mathcal{M}]}$ on the Jacobi group $SL(2,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)} \hookrightarrow G^J$ by

$$\Theta_f^{[\mathcal{M}]}(\tau,\phi\,;\lambda,\mu,\kappa) := \sum_{\omega\in\mathbb{Z}^{(m,n)}} \left[\pi_{\mathcal{M}}\left((\lambda,\mu;\kappa)(\tau,\phi)\right) f\right](\omega), \tag{12.22}$$

where $(\tau,\phi) \in SL(2,\mathbb{R})$ and $(\lambda,\mu\,;\kappa) \in H_{\mathbb{R}}^{(n,m)}$. The projective representation $\pi_{\mathcal{M}}$ of the Jacobi group G^J was already defined by Formula (12.8). More precisely, for $\tau = u+iv \in \mathbb{H}_1$ and $(\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)}$, we have

$$\begin{aligned}\Theta_f^{[\mathcal{M}]}(\tau,\phi\,;\lambda,\mu,\kappa) &= v^{\frac{mn}{4}}\ e^{2\,\pi\, i\,\sigma(\mathcal{M}(\kappa+\mu^t\lambda))} \\ &\times \sum_{\omega\in\mathbb{Z}^{(m,n)}} e^{\pi\, i\left\{u\|\omega+\lambda\|^2_{\mathcal{M}} + 2(\omega,\,\mu)_{\mathcal{M}}\right\}}\ \left[R_{\mathcal{M}}(i,\phi)f\right]\left(v^{1/2}(\omega+\lambda)\right).\end{aligned}$$

Lemma 12.2. *We set $f_\phi := \tilde{R}_{\mathcal{M}}(i,\phi)f$ for $f \in \mathscr{S}\left(\mathbb{R}^{(m,n)}\right)$. Then for any $R > 1$, there exists a constant C_R such that for all $x \in \mathbb{R}^{(m,n)}$ and $\phi \in \mathbb{R}$,*

$$|f_\phi(x)| \leq C_R\ \left(1+\|x\|_{\mathcal{M}}\right)^{-R}.$$

Proof. Following the arguments in the proof of Lemma 4.3 in [36], pp. 428–429, we get the desired result. □

Theorem 12.1 (Jacobi 1). *Let $\mathcal{M}$ be a positive definite symmetric integral matrix of degree m such that $\mathcal{M}\mathbb{Z}^{(m,n)} = \mathbb{Z}^{(m,n)}$. Then for any Schwartz function $f \in \mathscr{S}\left(\mathbb{R}^{(m,n)}\right)$, we have*

$$\Theta_f^{[\mathcal{M}]}\left(-\frac{1}{\tau},\, \phi + \arg\tau\,;-\mu,\lambda,\kappa\right) = \left(\det\mathcal{M}\right)^{-\frac{n}{2}} c_{\mathcal{M}}(S,(\tau,\phi))\,\Theta_f^{[\mathcal{M}]}(\tau,\phi\,;\lambda,\mu,\kappa),$$

where

$$c_{\mathcal{M}}(S,(\tau,\phi)) := e^{i\,\pi mn\, sign(\sin\phi\,\sin(\phi+\arg\tau))}.$$

Proof. See Theorem 6.1 in [78]. □

Theorem 12.2 (Jacobi 2). *Let $\mathcal{M} = (\mathcal{M}_{kl})$ be a positive definite symmetric integral $m\times m$ matrix and let $s = (s_{kj}) \in \mathbb{Z}^{(m,n)}$ be integral. Then we have*

$$\Theta_f^{[\mathcal{M}]}(\tau+2,\phi\,;\lambda, s-2\,\lambda+\mu, \kappa - s\,{}^t\lambda) = \Theta_f^{[\mathcal{M}]}(\tau,\phi\,;\lambda,\mu,\kappa)$$

for all $(\tau,\phi)\in SL(2,\mathbb{R})$ and $(\lambda,\mu;\kappa)\in H_{\mathbb{R}}^{(n,m)}$.

Proof. See Theorem 6.2 in [78]. □

Theorem 12.3 (Jacobi 3). *Let $\mathcal{M} = (\mathcal{M}_{kl})$ be a positive definite symmetric integral $m\times m$ matrix and let $(\lambda_0,\mu_0;\kappa_0)\in H_{\mathbb{Z}}^{(m,n)}$ be an integral element of $H_{\mathbb{R}}^{(n,m)}$. Then we have*

$$\begin{aligned}\Theta_f^{[\mathcal{M}]}&(\tau,\phi\,;\lambda+\lambda_0,\mu+\mu_0,\kappa+\kappa_0+\lambda_0\,{}^t\mu-\mu_0\,{}^t\lambda)\\ &= e^{\pi\, i\,\sigma(\mathcal{M}(\kappa_0+\mu_0\,{}^t\lambda_0))}\Theta_f^{[\mathcal{M}]}(\tau,\phi\,;\lambda,\mu,\kappa)\end{aligned}$$

for all $(\tau,\phi)\in SL(2,\mathbb{R})$ and $(\lambda,\mu;\kappa)\in H_{\mathbb{R}}^{(n,m)}$.

Proof. See Theorem 6.3 in [78]. □

We put $V(m,n) = \mathbb{R}^{(m,n)}\times\mathbb{R}^{(m,n)}$. Let

$$G^{(m,n)} := SL(2,\mathbb{R})\ltimes V(m,n)$$

be the group with the following multiplication law

$$(g_1,(\lambda_1,\mu_1))\cdot(g_2,(\lambda_2,\mu_2)) = (g_1g_2,(\lambda_1,\mu_1)g_2+(\lambda_2,\mu_2)), \tag{12.23}$$

where $g_1,g_2\in SL(2,\mathbb{R})$ and $\lambda_1,\lambda_2,\mu_1,\mu_2\in\mathbb{R}^{(m,n)}$.
We define

$$\Gamma^{(m,n)} := SL(2,\mathbb{Z})\times H_{\mathbb{Z}}^{(n,m)}.$$

Then $\Gamma^{(m,n)}$ acts on $G^{(m,n)}$ naturally through the multiplication law (12.23).

Lemma 12.3. *$\Gamma^{(m,n)}$ is generated by the elements*

$$(S,(0,0)),\quad (T_\flat,(0,s))\quad and\quad (I_2,(\lambda_0,\mu_0)),$$

where

$$S = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\quad T_\flat = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\quad and\ s,\lambda_0,\mu_0\in\mathbb{Z}^{(m,n)}.$$

Proof. Since $SL(2,\mathbb{Z})$ is generated by S and $T_\flat$, we get the desired result. □

We define

$$\Theta_f^{[\mathcal{M}]}(\tau,\phi;\lambda,\mu) = v^{\frac{mn}{4}} \sum_{\omega\in\mathbb{Z}^{(m,n)}} e^{\pi\, i\,\{u\|\omega+\lambda\|^2_{\mathcal{M}} + 2(\omega,\,\mu)_{\mathcal{M}}\}}\, [R_{\mathcal{M}}(i,\phi)f]\Big(v^{1/2}(\omega+\lambda)\Big).$$

Theorem 12.4. *Let $\Gamma_{[2]}^{(m,n)}$ be the subgroup of $\Gamma^{(m,n)}$ generated by the elements*

$$(S,(0,0)),\quad (T_*,(0,s))\quad \text{and}\quad (I_2,(\lambda_0,\mu_0)),$$

where

$$T_* = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad s,\lambda_0,\mu_0 \in \mathbb{Z}^{(m,n)}.$$

Let $\mathcal{M} = (\mathcal{M}_{kl})$ be a positive definite symmetric unimodular integral $m\times m$ matrix such that $\mathcal{M}\mathbb{Z}^{(m,n)} = \mathbb{Z}^{(m,n)}$. Then for $f,g \in \mathscr{S}\left(\mathbb{R}^{(m,n)}\right)$, the function

$$\Theta_f^{[\mathcal{M}]}(\tau,\phi;\lambda,\mu)\,\overline{\Theta_g^{[\mathcal{M}]}(\tau,\phi;\lambda,\mu)}$$

is invariant under the action of $\Gamma_{[2]}^{(m,n)}$ on $G^{(m,n)}$.

Proof. See Theorem 6.4 in [78]. □

13. Final remarks and open problems

The Siegel–Jacobi space $\mathbb{H}_{n,m}$ is a non-symmetric homogeneous space that is important geometrically and arithmetically. As we see in Formula (7.2), the theory of Jacobi forms is applied in the study of modular forms. The theory of Jacobi forms reduces to that of Siegel modular forms if the index $\mathcal{M}$ is zero. Unfortunately the theory of the geometry and the arithmetic of the Siegel–Jacobi space has not been well developed so far.

Now we propose open problems related to the geometry and the arithmetic of the Siegel–Jacobi space.

Problem 1. Find the analogue of the Hirzebruch–Mumford Proportionality Theorem.

Let us give some remarks for this problem. Before we describe the proportionality theorem for the Siegel modular variety, first of all we review the compact dual of the Siegel upper half-plane $\mathbb{H}_n$. We note that $\mathbb{H}_n$ is biholomorphic to the generalized unit disk $\mathbb{D}_n$ of degree n through the Cayley transform. We suppose that $\Lambda = (\mathbb{Z}^{2n}, \langle\ ,\ \rangle)$ is a symplectic lattice with a symplectic form $\langle\ ,\ \rangle$. We extend scalars of the lattice Λ to $\mathbb{C}$. Let

$$\mathfrak{Y}_n := \left\{\, L \subset \mathbb{C}^{2n} \mid\ \dim_{\mathbb{C}} L = n,\ \ \langle x,y\rangle = 0 \quad \text{for all } x,y\in L \,\right\}$$

be the complex Lagrangian Grassmannian variety parameterizing totally isotropic subspaces of complex dimension n. For the present time being, for brevity, we put $G = Sp(n,\mathbb{R})$ and $K = U(n)$. The complexification $G_{\mathbb{C}} = Sp(n,\mathbb{C})$ of G acts on

$\mathfrak{Y}_n$ transitively. If H is the isotropy subgroup of $G_{\mathbb{C}}$ fixing the first summand $\mathbb{C}^n$, we can identify $\mathfrak{Y}_n$ with the compact homogeneous space $G_{\mathbb{C}}/H$. We let

$$\mathfrak{Y}_n^+ := \left\{ L \in \mathfrak{Y}_n \mid -i\langle x, \bar{x} \rangle > 0 \quad \text{for all } x(\neq 0) \in L \right\}$$

be an open subset of $\mathfrak{Y}_n$. We see that G acts on $\mathfrak{Y}_n^+$ transitively. It can be shown that $\mathfrak{Y}_n^+$ is biholomorphic to $G/K \cong \mathbb{H}_n$. A basis of a lattice $L \in \mathfrak{Y}_n^+$ is given by a unique $2n \times n$ matrix ${}^t(-I_n\ \Omega)$ with $\Omega \in \mathbb{H}_n$. Therefore we can identify L with Ω in $\mathbb{H}_n$. In this way, we embed $\mathbb{H}_n$ into $\mathfrak{Y}_n$ as an open subset of $\mathfrak{Y}_n$. The complex projective variety $\mathfrak{Y}_n$ is called the *compact dual* of $\mathbb{H}_n$.

Let Γ be an arithmetic subgroup of Γ_n. Let E_0 be a G-equivariant holomorphic vector bundle over $\mathbb{H}_n = G/K$ of rank r. Then E_0 is defined by the representation $\tau : K \longrightarrow GL(r, \mathbb{C})$. That is, $E_0 \cong G \times_K \mathbb{C}^r$ is a homogeneous vector bundle over G/K. We naturally obtain a holomorphic vector bundle E over $\mathcal{A}_{n,\Gamma} := \Gamma\backslash G/K$. E is often called an *automorphic* or *arithmetic* vector bundle over $\mathcal{A}_{n,\Gamma}$. Since K is compact, E_0 carries a G-equivariant Hermitian metric h_0 which induces a Hermitian metric h on E. According to Main Theorem in [43], E admits a *unique* extension $\tilde{E}$ to a smooth toroidal compactification $\tilde{\mathcal{A}}_{n,\Gamma}$ of $\mathcal{A}_{n,\Gamma}$ such that h is a singular Hermitian metric *good* on $\tilde{\mathcal{A}}_{n,\Gamma}$. For the precise definition of a *good metric* on $\mathcal{A}_{n,\Gamma}$ we refer to [43, p. 242]. According to Hirzebruch–Mumford's Proportionality Theorem (cf. [43, p. 262]), there is a natural metric on $G/K = \mathbb{H}_n$ such that the Chern numbers satisfy the following relation

$$c^{\alpha}\left(\tilde{E}\right) = (-1)^{\frac{1}{2}n(n+1)} \operatorname{vol}\left(\Gamma\backslash\mathbb{H}_n\right)\, c^{\alpha}\left(\check{E}_0\right)$$

for all $\alpha = (\alpha_1, \ldots, \alpha_r)$ with nonnegative integers $\alpha_i\ (1 \le i \le r)$ and $\sum_{i=1}^{r} \alpha_i = \frac{1}{2}n(n+1)$, where $\check{E}_0$ is the $G_{\mathbb{C}}$-equivariant holomorphic vector bundle on the compact dual $\mathfrak{Y}_n$ of $\mathbb{H}_n$ defined by a certain representation of the stabilizer $\mathrm{Stab}_{G_{\mathbb{C}}}(e)$ of a point e in $\mathfrak{Y}_n$. Here $\operatorname{vol}(\Gamma\backslash\mathbb{H}_n)$ is the volume of $\Gamma\backslash\mathbb{H}_n$ that can be computed (cf. [57]).

As before we consider the Siegel–Jacobi modular group $\Gamma_{n.m} := \Gamma_n \ltimes H_{\mathbb{Z}}^{(n,m)}$ with $\Gamma_n = Sp(n, \mathbb{Z})$. For an arithmetic subgroup Γ of Γ_n, we set

$$\mathcal{A}_{n,m,\Gamma} := \Gamma_* \backslash \mathbb{H}_{n,m} \qquad \text{with } \Gamma_* = \Gamma \ltimes H_{\mathbb{Z}}^{(n,m)}.$$

Problem 2. Compute the cohomology $H^{\bullet}(\mathcal{A}_{n,m,\Gamma}, *)$ of $\mathcal{A}_{n,m,\Gamma}$. Investigate the intersection cohomology of $\mathcal{A}_{n,m,\Gamma}$.

Problem 3. Generalize the trace formula on the Siegel modular variety obtained by Sophie Morel to the universal Abelian variety. For her result on the trace formula on the Siegel modular variety, we refer to her paper, *Cohomologie d'intersection des variétés modulaires de Siegel, suite.*

Problem 4. Develop the theory of the stability of Jacobi forms using the Siegel–Jacobi operator. The theory of the stability involves in the theory of unitary representations of the infinite-dimensional symplectic group $Sp(\infty, \mathbb{R})$ and the infinite-dimensional unitary group $U(\infty)$.

Problem 5. Compute the geodesics, the distance between two points and curvatures explicitly in the Siegel–Jacobi space $(\mathbb{H}_{n,m}, ds^2_{n,m;A,B})$.

Siegel proved the following theorem for the Siegel space $(\mathbb{H}_n, ds^2_{n;1})$.

Theorem 13.1 (Siegel [57]).

(1) *There exists exactly one geodesic joining two arbitrary points* Ω_0, Ω_1 *in* $\mathbb{H}_n$. *Let* $R(\Omega_0, \Omega_1)$ *be the cross-ratio defined by*

$$R(\Omega_0, \Omega_1) = (\Omega_0 - \Omega_1)(\Omega_0 - \overline{\Omega}_1)^{-1}(\overline{\Omega}_0 - \overline{\Omega}_1)(\overline{\Omega}_0 - \Omega_1)^{-1}.$$

For brevity, we put $R_* = R(\Omega_0, \Omega_1)$. *Then the symplectic length* $\rho(\Omega_0, \Omega_1)$ *of the geodesic joining* Ω_0 *and* Ω_1 *is given by*

$$\rho(\Omega_0, \Omega_1)^2 = \sigma\left(\left(\log \frac{1+R_*^{\frac{1}{2}}}{1-R_*^{\frac{1}{2}}}\right)^2\right),$$

where

$$\left(\log \frac{1+R_*^{\frac{1}{2}}}{1-R_*^{\frac{1}{2}}}\right)^2 = 4\,R_*\left(\sum_{k=0}^{\infty}\frac{R_*^k}{2k+1}\right)^2.$$

(2) *For* $M \in Sp(n, \mathbb{R})$, *we set*

$$\tilde{\Omega}_0 = M\cdot\Omega_0 \quad \textit{and} \quad \tilde{\Omega}_1 = M\cdot\Omega_1.$$

Then $R(\Omega_1, \Omega_0)$ *and* $R(\tilde{\Omega}_1, \tilde{\Omega}_0)$ *have the same eigenvalues.*

(3) *All geodesics are symplectic images of the special geodesics*

$$\alpha(t) = i\,\mathrm{diag}\,(a_1^t, a_2^t, \ldots, a_n^t),$$

where $a_1, a_2, \ldots, a_n$ *are arbitrary positive real numbers satisfying the condition*

$$\sum_{k=1}^{n}(\log a_k)^2 = 1.$$

The proof of the above theorem can be found in [57], pp. 289–293.

Problem 6. Solve Problem 4 and Problem 5 in Section 3. Express the center of the algebra $\mathbb{D}(\mathbb{H}_{n,m})$ of all G^J-invariant differential operators on $\mathbb{H}_{n,m}$ explicitly. Describe the center of the universal enveloping algebra of the Lie algebra of the Jacobi group G^J explicitly.

Problem 7. Develop the spectral theory of the Laplacian $\Delta_{n,m;A,B}$ on $\Gamma_*\backslash\mathbb{H}_{n,m}$ for an arithmetic subgroup of $\Gamma_{n,m}$. Balslev [2] developed the spectral theory of the Laplacian $\Delta_{1,1;1,1}$ on $\Gamma_*\backslash\mathbb{H}_{1,1}$ for certain arithmetic subgroup of $\Gamma_{1,1}$.

Problem 8. Develop the theory of harmonic analysis on the Siegel–Jacobi disk $\mathbb{H}_{n,m}$.

Problem 9. Study unitary representations of the Jacobi group G^J. Develop the theory of the orbit method for the Jacobi group G^J.

Problem 10. Attach Galois representations to cuspidal Jacobi forms.

Problem 11. Develop the theory of automorphic L-function for the Jacobi group $G^J(\mathbb{A})$.

Problem 12. Find the trace formula for the Jacobi group $G^J(\mathbb{A})$.

Problem 13. Decompose the Hilbert space $L^2\big(G^J(\mathbb{Q})\backslash G^J(\mathbb{A})\big)$ into irreducibles explicitly.

Problem 14. Construct Maass–Jacobi forms. Express the Fourier expansion of a Maass–Jacobi form explicitly.

Problem 15. Investigate the relations among Jacobi forms, hyperbolic Kac–Moody algebras, infinite products, the monster group and the Moonshine (cf. [67]).

Problem 16. Provide applications to physics (quantum mechanics, quantum optics, coherent states,. . .), the theory of elliptic genera, singularity theory of K. Saito etc.

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