Problems in Invariant Differential Operators on Homogeneous Manifolds

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October 26 (Sat), 2024
2024 KMS Annual Meeting

Sungkyunkwan University
Natural Sciences Campus (Suwon)

Table of Contents

Homogeneous Manifolds

We consider the following six homogeneous manifolds which are important geometrically and number theoretically.

- $GL(n,\mathbb{R})/O(n,\mathbb{R})$
- $SL(n,\mathbb{R})/SO(n,\mathbb{R})$
- $Sp(2n,\mathbb{R})/U(n)$
- $\left(GL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}\right)/O(n,\mathbb{R})$
- $\left(SL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}\right)/SO(n,\mathbb{R})$
- $\left(Sp(2n,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)}\right)/(U(n) \times S(m,\mathbb{R}))$

II. Invariant Differential Operators on $GL(n,\mathbb{R})/O(n,\mathbb{R})$

$GL(n,\mathbb{R})/O(n,\mathbb{R})$

- For any positive integer $n \geq 1$, we let
$$\mathscr{P}_n := \{ Y \in \mathbb{R}^{(n,n)} \mid Y = {}^tY > 0 \}$$
be the open convex cone in the Euclidean space $\mathbb{R}^N$ with $N = \frac{n(n+1)}{2}$.
- $GL(n,\mathbb{R})$ **acts** $\mathscr{P}_n$ **transitively** by
$$g \cdot Y = gY\,{}^tg, \tag{2.1}$$
where $g \in GL(n,\mathbb{R})$ and $Y \in \mathscr{P}_n$.
- Since $O(n)$ is the isotopic subgroup of $GL(n,\mathbb{R})$ at I_n, the symmetric space $GL(n,\mathbb{R})/O(n)$ **is diffeomorphic to** $\mathscr{P}_n$.

$GL(n,\mathbb{R})/O(n,\mathbb{R})$

- For $Y = (y_{ij}) \in \mathscr{P}_n$, we put

$$dY = (dy_{ij}) \qquad \text{and} \qquad \frac{\partial}{\partial Y} = \left(\frac{1+\delta_{ij}}{2} \frac{\partial}{\partial y_{ij}} \right).$$

- For any positive real number $C > 0$,

$$ds^2_{n;C} = C \cdot \mathrm{Tr}\left((Y^{-1}dY)^2\right)$$

is a **Riemannian metric on $\mathscr{P}_n$ invariant under the action (2.1).**

- **Laplace operator** is given by

$$\Delta_{n;C} = \frac{1}{C} \cdot \mathrm{Tr}\left(\left(Y \frac{\partial}{\partial Y} \right)^2 \right),$$

where $\mathrm{Tr}(M)$ denotes the trace of a square matrix M.

Invariant Differential Operators on $GL(n,\mathbb{R})/O(n,\mathbb{R})$

We consider the following **Maass-Selberg (differential) operators** $\delta_1, \delta_2, \cdots, \delta_n$ on $\mathscr{P}_n$ defined by

$$\delta_k = \mathrm{Tr}\left(\left(Y\frac{\partial}{\partial Y}\right)^k\right), \quad k = 1, 2, \cdots, n. \tag{2.2}$$

Each δ_i $(1 \le i \le n)$ is **invariant under the action** (2.1) of $GL(n,\mathbb{R})$.

Theorem 1 (Maass and Selberg)

*The algebra $\mathbb{D}(\mathscr{P}_n)$ of all $GL(n,\mathbb{R})$-invariant differential operators on $\mathscr{P}_n$ is **generated by** $\delta_1, \delta_2, \cdots, \delta_n$.*
*Furthermore, $\delta_1, \delta_2, \cdots, \delta_n$ are **algebraically independent** and $\mathbb{D}(\mathscr{P}_n)$ is **isomorphic to** the commutative ring $\mathbb{C}[x_1, x_2, \cdots, x_n]$ with n indeterminates $x_1, x_2, \cdots, x_n$.*

Remark

A different description of $\mathbb{D}(\mathscr{P}_n)$ was given by Helgason.

III. Invariant Differential Operators on $SL(n,\mathbb{R})/SO(n,\mathbb{R})$

$SL(n,\mathbb{R})/SO(n,\mathbb{R})$

- Let $\mathfrak{P}_n := \left\{ Y \in \mathbb{R}^{(n,n)} \mid Y = {}^tY > 0,\ \det(Y) = 1 \right\}$ be a symmetric space associated to $SL(n,\mathbb{R})$.
- $SL(n,\mathbb{R})$ **acts on $\mathfrak{P}_n$ transitively** by

$$g \circ Y = gY\,{}^tg, \qquad g \in SL(n,\mathbb{R}),\ Y \in \mathfrak{P}_n. \tag{3.1}$$

- $\mathfrak{P}_n$ is a smooth manifold **diffeomorphic to** the symmetric space $SL(n,\mathbb{R})/SO(n,\mathbb{R})$ through the bijective map

$$g \cdot SO(n,\mathbb{R}) \mapsto g \circ I_n = g\,{}^tg, \quad g \in SL(n,\mathbb{R}).$$

- Let $\mathbb{D}(\mathfrak{P}_n)$ be **the algebra of all differential operators on $\mathfrak{P}_n$ invariant under the action (3.1)** of $SL(n,\mathbb{R})$.
- $\mathbb{D}(\mathfrak{P}_n)$ **is isomorphic to** the polynomial algebra $\mathbb{C}[x_1, x_2, \cdots, x_{n-1}]$ with $n-1$ indeterminates $x_1, x_2, \cdots, x_{n-1}$.
- $n-1$ is **the rank of the symmetric space $SL(n,\mathbb{R})/SO(n,\mathbb{R})$**.

Invariant Differential Operators on $SL(n,\mathbb{R})/SO(n,\mathbb{R})$

Theorem 2 (Brennecken, Ciardo and Hilgert, 2020)

Let $\delta_1, \delta_2, \ldots, \delta_n$ be the Maass-Selberg operators, and consider the mapping $\mathscr{L} : \mathbb{D}(GL(n,\mathbb{R})/O(n,\mathbb{R})) \longrightarrow \mathbb{D}(SL(n,\mathbb{R})/SO(n,\mathbb{R}))$ defined by $\mathscr{L}(\delta_1) = 0$, and for $2 \leq k \leq n$ by

$$\mathscr{L}(\delta_k) f(g \cdot SO(n,\mathbb{R})) := \delta_k|_{X=0} f\left((g \cdot \exp(X - n^{-1}\mathrm{Tr}(X) I_n)) \cdot SO(n,\mathbb{R})\right)$$

for all $f \in C^\infty(SL(n,\mathbb{R})/SO(n,\mathbb{R}))$, where $X = (x_{ij}) \in \mathbb{R}^{(n,n)}$ is a symmetric matrix and $\frac{\partial}{\partial X} = \left(\frac{1+\delta_{ij}}{2} \frac{\partial}{\partial x_{ij}}\right)$.
*Then, the differential operators $\mathscr{L}(\delta_2), \mathscr{L}(\delta_3), \cdots, \mathscr{L}(\delta_n)$ **are algebraically independent generators** of $\mathbb{D}(SL(n,\mathbb{R})/SO(n,\mathbb{R}))$.*

IV. Invariant Differential Operators on $Sp(2n,\mathbb{R})/U(n)$

$Sp(2n,\mathbb{R})/U(n)$

- Let $G := Sp(2n,\mathbb{R}), K := U(n)$ and

$$\mathbb{H}_n := \{\, \Omega \in \mathbb{C}^{(n,n)} \mid \Omega = {}^t\Omega,\ \ \mathrm{Im}\,\Omega > 0 \,\}$$

  be the **Siegel upper half plane** of degree n.

- ***G* acts on $\mathbb{H}_n$ transitively** by

$$M \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1}, \tag{4.1}$$

  where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$ and $\Omega \in \mathbb{H}_n$.

- The **stabilizer** of the action (4.1) at iI_n is

$$K = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \,\middle|\, A + iB \in U(n) \right\} \cong U(n).$$

- Thus we get the biholomorphic map $G/K \longrightarrow \mathbb{H}_n$ given by

$$gK \mapsto g\cdot iI_n, \quad g \in G.$$

  $\mathbb{H}_n$ is an **Einstein-Kähler Hermitian symmetric manifold.**

$Sp(2n,\mathbb{R})/U(n)$

For $\Omega = (\omega_{ij}) \in \mathbb{H}_n$, we write $\Omega = X + i\,Y$ with $X = (x_{ij}),\ Y = (y_{ij})$ real. We put $d\Omega = (d\omega_{ij})$ and $d\overline{\Omega} = (d\overline{\omega}_{ij})$. We also put

$$\frac{\partial}{\partial\Omega} = \left(\frac{1+\delta_{ij}}{2}\, \frac{\partial}{\partial\omega_{ij}} \right) \qquad \text{and} \qquad \frac{\partial}{\partial\overline{\Omega}_{ij}} = \left(\frac{1+\delta_{ij}}{2}\, \frac{\partial}{\partial\overline{\omega}_{ij}} \right).$$

- C. L. Siegel (1943) introduced the **symplectic metric $ds^2_{n;A}$ on $\mathbb{H}_n$ invariant under the action (4.1)** of $Sp(2n,\mathbb{R})$ that is given by

$$ds^2_{n;A} = A \cdot \mathrm{Tr}(Y^{-1} d\Omega\, Y^{-1} d\overline{\Omega}), \qquad A > 0.$$

- H. Maass (1953) proved that its **Laplace operator** $\Delta_{n;A}$ is given by

$$\Delta_{n;A} = \frac{4}{A} \cdot \mathrm{Tr} \left(Y\, {}^t\!\left(Y \frac{\partial}{\partial\overline{\Omega}} \right) \frac{\partial}{\partial\Omega} \right).$$

And

$$dv_n(\Omega) = (\det Y)^{-(n+1)} \prod_{1\le i\le j\le n} dx_{ij} \prod_{1\le i\le j\le n} dy_{ij}$$

is a $Sp(2n,\mathbb{R})$-**invariant volume element** on $\mathbb{H}_n$.

Invariant Differential Operators on $Sp(2n,\mathbb{R})/U(n)$

- Let $\mathbb{D}(\mathbb{H}_n)$ be the algebra of **all differential operators** on $\mathbb{H}_n$ invariant under the action (4.1).
- According to Harish-Chandra (1956), $\mathbb{D}(\mathbb{H}_n)$ is a commutative algebra, **finitely generated by n algebraically independent invariant differential operators** $D_1, \cdots, D_n$ on $\mathbb{H}_n$.
- Maass (1971) found the explicit expressions for $D_1, \cdots, D_n$.
- Let T_n be the vector space of $n \times n$ symmetric complex matrices. And we denote by $\mathrm{Pol}(T_n)^{U(n)}$ the subalgebra of the polynomial algebra $\mathrm{Pol}(T_n)$ consisting of **all $U(n)$-invariants** with respect to the action induced by the adjoint representation.
- We get a canonical linear bijection

$$\mathscr{H}_{C,n} : \mathrm{Pol}(T_n)^{U(n)} \longrightarrow \mathbb{D}(\mathbb{H}_n)$$

of $\mathrm{Pol}(T_n)^{U(n)}$ onto $\mathbb{D}(\mathbb{H}_n)$.

Example 3 ($n = 1$ case)

The algebra $\mathrm{Pol}(T_1)^{U(1)}$ is generated by the polynomial

$$q(\omega) = \omega\,\overline{\omega}, \quad \omega = x + iy \in \mathbb{C} \text{ with } x, y \text{ real.}$$

Moreover, we get

$$\mathscr{H}_{C,1}(q) = 4\,y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

Therefore $\mathbb{D}(\mathbb{H}_1) = \mathbb{C}\big[\mathscr{H}_{C,1}(q)\big]$.

In fact, we see that

$$\Delta_{n;1} = 4\,\mathrm{Tr}\left(Y \,{}^t\!\left(Y \frac{\partial}{\partial \overline{\Omega}} \right) \frac{\partial}{\partial \Omega} \right).$$

is the **Laplace operator** for the invariant metric $ds^2_{n;1}$ on $\mathbb{H}_n$.

V. Invariant Differential Operators on $\left(GL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}\right)/O(n,\mathbb{R})$

$\left(GL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}\right)/O(n,\mathbb{R})$

- The group $GL_{n,m}(\mathbb{R}) := GL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}$ is the semidirect product of $GL(n,\mathbb{R})$ and the additive group $\mathbb{R}^{(m,n)}$ endowed with **multiplication law**

$$(g,\alpha)\circ(h,\beta) := (gh, \alpha\,{}^th^{-1} + \beta) \tag{5.1}$$

  for all $g,h \in GL(n,\mathbb{R})$ and $\alpha,\beta \in \mathbb{R}^{(m,n)}$.

- $GL_{n,m}(\mathbb{R})$ **acts on** the Minkowski-Euclid space $\mathscr{P}_{n,m} := \mathscr{P}_n \times \mathbb{R}^{(m,n)}$ naturally and **transitively** by

$$(g,\alpha)\cdot(Y,V) := (gY\,{}^tg, (V+\alpha)\,{}^tg) \tag{5.2}$$

  for all $(g,\alpha) \in GL_{n,m}(\mathbb{R})$ and $(Y,V) \in \mathscr{P}_{n,m}$.

- Since $O(n,\mathbb{R})$ is the stabilizer of the action (5.1) at $(I_n,0)$, the **non-symmetric homogeneous space $GL_{n,m}(\mathbb{R})/O(n,\mathbb{R})$ is diffeomorphic to the Minkowski-Euclid space $\mathscr{P}_{n,m}$.**

Lemma 4

For all two positive real numbers a and b, the following metric $ds^2_{n,m;a,b}$ on $\mathscr{P}_{n,m}$ defined by

$$ds^2_{n,m;a,b} = a \cdot \mathrm{Tr}(Y^{-1}dY\,Y^{-1}dY) + b \cdot \mathrm{Tr}(Y^{-1}\,{}^t(dV)\,dV)$$

is a ***Riemannian metric on $\mathscr{P}_{n,m}$ which is invariant under the action*** (5.1) *of $GL_{n,m}(\mathbb{R})$.*
The Laplacian $\Delta_{n,m;a,b}$ of $(\mathscr{P}_{n,m},\, ds^2_{n,m;a,b})$ is given by

$$\frac{1}{a} \cdot \mathrm{Tr}\left(\left(Y\frac{\partial}{\partial Y}\right)^2\right) - \frac{m}{2\,a}\,\mathrm{Tr}\left(Y\frac{\partial}{\partial Y}\right) + \frac{1}{b} \cdot \sum_{k\leq p}\left(\left(\frac{\partial}{\partial V}\right)Y\,{}^t\!\left(\frac{\partial}{\partial V}\right)\right)_{kp}.$$

Moreover, ***the Laplacian $\triangle_{n,m;a,b}$ is a differential operator of order 2 which is invariant under the action*** (5.1) *of $GL_{n,m}(\mathbb{R})$.*

Invariant Differential Operators on $GL_{n,m}(\mathbb{R})/O(n,\mathbb{R})$

- The **Lie algebra** $\mathfrak{g}_\star$ of $GL_{n,m}(\mathbb{R})$ is given by

$$\mathfrak{g}_\star = \Big\{ (X,Z) \,|\, X \in \mathbb{R}^{(n,n)},\ Z \in \mathbb{R}^{(m,n)} \Big\}$$

equipped with the following Lie bracket

$$\big[(X_1,Z_1),\,(X_2,Z_2)\big]_\star = \big([X_1,X_2]_0,\, Z_2\,{}^tX_1 - Z_1\,{}^tX_2\big),$$

where $[X_1,X_2]_0 = X_1X_2 - X_2X_1$ denotes the usual matrix bracket and $(X_1,Z_1),\,(X_2,Z_2) \in \mathfrak{g}_\star$.

- Let $K_\star := \{\,(k,0) \in GL_{n,m}(\mathbb{R}) \,|\, k \in O(n,\mathbb{R})\,\} \cong K := O(n,\mathbb{R})$. Then the Lie algebra $\mathfrak{k}_\star$ of $K_\star$ is

$$\mathfrak{k}_\star = \Big\{ (X,0) \in \mathfrak{g}_\star \,\Big|\, X + {}^tX = 0 \Big\}.$$

We let $\mathfrak{p}_\star$ be the subspace of $\mathfrak{g}_\star$ defined by

$$\mathfrak{p}_\star = \Big\{ (X,Z) \in \mathfrak{g}_\star \,\Big|\, X = {}^tX \in \mathbb{R}^{(n,n)},\ Z \in \mathbb{R}^{(m,n)} \Big\}.$$

Then we have $\mathfrak{g}_\star = \mathfrak{k}_\star \oplus \mathfrak{p}_\star$ **(the direct sum)**.

- **$K_\star$ acts on $\mathfrak{p}_\star$ via the adjoint representation** of $GL_{n,m}(\mathbb{R})$ by

$$k_\star \cdot (X, Z) = \big(kX\,{}^t k, Z\,{}^t k\big), \tag{5.3}$$

where $k_\star = (k, 0) \in K_\star$ with $k \in O(n, \mathbb{R})$ and $(X, Z) \in \mathfrak{p}_\star$.

- The action (5.3) induces **the action of K on the polynomial algebra $\mathrm{Pol}(\mathfrak{p}_\star)$ of $\mathfrak{p}_\star$** and the symmetric algebra $S(\mathfrak{p}_\star)$. We denote by $\mathrm{Pol}(\mathfrak{p}_\star)^K$ (resp. $S(\mathfrak{p}_\star)^K$) the subalgebra of $\mathrm{Pol}(\mathfrak{p}_\star)$ (resp. $S(\mathfrak{p}_\star)$) consisting of all **K-invariants.**
- We denote by $\mathbb{D}(\mathscr{P}_{n,m})$ the algebra of **all differential operators on $\mathscr{P}_{n,m}$ invariant under the action (5.1)** of $GL_{n,m}(\mathbb{R})$. It is known that there is a canonical linear bijection of $S(\mathfrak{p}_\star)^K$ onto $\mathbb{D}(\mathscr{P}_{n,m})$. Then we get a canonical linear bijection of $\mathrm{Pol}(\mathfrak{p}_\star)^K$ onto $\mathbb{D}(\mathscr{P}_{n,m})$

$$\Phi_{n,m} : \mathrm{Pol}(\mathfrak{p}_\star)^K \longrightarrow \mathbb{D}(\mathscr{P}_{n,m}). \tag{5.4}$$

Problems in $\mathbb{D}(\mathscr{P}_{n,m})$

We propose the following natural problems.

1. Find a **complete list** of explicit generators of $\mathrm{Pol}(\mathfrak{p}_\star)^K$.
2. Find all the **relations** among a set of generators of $\mathrm{Pol}(\mathfrak{p}_\star)^K$.
3. Find an easy or effective way to express the images of the above invariant polynomials under the **Helgason map** $\Phi_{n,m}$ explicitly.
4. Decompose $\mathrm{Pol}(\mathfrak{p}_\star)^K$ into $O(n,\mathbb{R})$**-irreducibles**.
5. Find a **complete list** of explicit generators of the algebra $\mathbb{D}(\mathscr{P}_{n,m})$. Or construct explicit $GL_{n,m}(\mathbb{R})$-invariant differential operators on $\mathscr{P}_{n,m}$.
6. Find all the **relations** among a set of generators of $\mathbb{D}(\mathscr{P}_{n,m})$.
7. Is $\mathrm{Pol}(\mathfrak{p}_\star)^K$ **finitely generated** ? Is $\mathbb{D}(\mathscr{P}_{n,m})$ **finitely generated** ?

Minoru Itoh (2013) solved Problem 1 and Problem 7.

Remark

$\mathbb{D}(\mathscr{P}_{n,m})$ is **not commutative**.

VI. Invariant Differential Operators on $\left(SL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}\right)/SO(n,\mathbb{R})$

$SL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}/SO(n,\mathbb{R})$

- The group $SL_{n,m}(\mathbb{R}) := SL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}$ is the semidirect product of $SL(n,\mathbb{R})$ and the additive group $\mathbb{R}^{(m,n)}$ with the multiplication law

$$(g,\alpha) \circ (h,\beta) := (gh, \alpha\, {}^t h^{-1} + \beta) \tag{6.1}$$

  for all $g, h \in SL(n,\mathbb{R})$ and $\alpha, \beta \in \mathbb{R}^{(m,n)}$.

- $SL_{n,m}(\mathbb{R})$ **acts on** the space $\mathfrak{P}_{n,m} := \mathfrak{P}_n \times \mathbb{R}^{(m,n)}$ naturally and **transitively** by

$$(g,\alpha) \cdot (Y,V) := (gY\, {}^t g, (V+\alpha)\, {}^t g) \tag{6.2}$$

  for all $(g,\alpha) \in SL_{n,m}(\mathbb{R})$ and $(Y,V) \in \mathfrak{P}_{n,m}$.

- Since $SO(n,\mathbb{R})$ is the stabilizer of the action (6.2) at $(I_n, 0)$, the **non-symmetric homogeneous space $SL_{n,m}(\mathbb{R})/SO(n,\mathbb{R})$ is diffeomorphic to the space $\mathfrak{P}_{n,m}$.**

Invariant Differential Operators on $SL_{n,m}(\mathbb{R})/SO(n,\mathbb{R})$

- The **Lie algebra $\mathfrak{g}_\diamond$** of $GL_{n,m}(\mathbb{R})$ is given by

$$\mathfrak{g}_\diamond = \left\{ (X,Z) \mid X \in \mathfrak{sl}(n,\mathbb{R}),\ Z \in \mathbb{R}^{(m,n)} \right\} \qquad (6.3)$$

equipped with the following Lie bracket

$$[(X_1,Z_1),\,(X_2,Z_2)]_\diamond = ([X_1,X_2]_0,\, Z_2\,{}^tX_1 - Z_1\,{}^tX_2), \qquad (6.4)$$

where $[X_1,X_2]_0 := X_1X_2 - X_2X_1$ denotes the usual matrix bracket and $(X_1,Z_1),\,(X_2,Z_2) \in \mathfrak{g}_\diamond$.

- Let $K_\diamond := \{\,(k,0) \in GL_{n,m}(\mathbb{R}) \mid k \in SO(n,\mathbb{R})\,\} \cong K_\natural := SO(n,\mathbb{R})$. Then the Lie algebra $\mathfrak{k}_\diamond$ of $K_\diamond$ is

$$\mathfrak{k}_\diamond = \{\,(X,0) \in \mathfrak{g}_\diamond \,\big|\, X + {}^tX = 0,\ X \in \mathbb{R}^{(n,n)},\ 0 \in \mathbb{R}^{(m,n)}\,\}.$$

We let $\mathfrak{p}_\diamond$ be the subspace of $\mathfrak{g}_\diamond$ defined by

$$\mathfrak{p}_\diamond = \left\{ (X,Z) \in \mathfrak{g}_\diamond \mid X = {}^tX \in \mathbb{R}^{(n,n)},\ \mathrm{Tr}(X) = 0,\ Z \in \mathbb{R}^{(m,n)} \right\}.$$

Then we have $\mathfrak{g}_\diamond = \mathfrak{k}_\diamond \oplus \mathfrak{p}_\diamond$ **(direct sum)**.

- $K_\diamond$ **acts on** $\mathfrak{p}_\diamond$ **via the adjoint representation** of $K_\diamond$ on $\mathfrak{p}_\diamond$ by

$$k_\diamond \cdot (X, Z) = \big(kX\,{}^t k, Z\,{}^t k\big), \tag{6.5}$$

where $k_\diamond = (k, 0) \in K_\diamond$ with $k \in SO(n, \mathbb{R})$ and $(X, Z) \in \mathfrak{p}_\diamond$.

- The action (6.5) induces the action of $K_\natural$ on the polynomial algebra $\mathrm{Pol}(\mathfrak{p}_\diamond)$ of $\mathfrak{p}_\diamond$ and the symmetric algebra $S(\mathfrak{p}_\diamond)$. We denote by $\mathbf{Pol}(\mathfrak{p}_\diamond)^{K_\natural}$ **(resp.** $S(\mathfrak{p}_\diamond)^{K_\natural}$**)** the subalgebra of $\mathrm{Pol}(\mathfrak{p}_\diamond)$ (resp. $S(\mathfrak{p}_\diamond)$) consisting of all $K_\natural$**-invariants**.

- The following **inner product $(\ ,\ )_\diamond$ on $\mathfrak{p}_\diamond$** defined by

$$\big((X_1, Z_1), (X_2, Z_2)\big)_\diamond = \mathrm{Tr}(X_1 X_2) + \mathrm{Tr}(Z_1\, {}^t Z_2),$$

for $(X_1, Z_1), (X_2, Y_2) \in \mathfrak{p}_\diamond$ gives an isomorphism as vector spaces

$$\mathfrak{p}_\diamond \cong \mathfrak{p}_\diamond^*, \tag{6.6}$$

- Let $\mathbb{D}(\mathfrak{P}_{n,m})$ be the algebra of **all differential operators on $\mathfrak{P}_{n,m}$ that are invariant under the action (6.2)** of $GL_{n,m}(\mathbb{R})$.
- It is known that there is a canonical linear bijection of $S(\mathfrak{p}_\diamond)^K$ onto $\mathbb{D}(\mathfrak{P}_{n,m})$. Identifying $\mathfrak{p}_\diamond$ with $\mathfrak{p}_\diamond^*$ by the above isomorphism (6.6), we get a **canonical linear bijection**

$$\Psi_{n,m} : \mathrm{Pol}(\mathfrak{p}_\diamond)^{K_\natural} \longrightarrow \mathbb{D}(\mathfrak{P}_{n,m})$$

of $\mathrm{Pol}(\mathfrak{p}_\diamond)^{K_\natural}$ onto $\mathbb{D}(\mathfrak{P}_{n,m})$.

Problems in $\mathbb{D}(\mathfrak{P}_{n,m})$

We propose the following problems:

1. Find a **complete list** of explicit generators of $\mathrm{Pol}(\mathfrak{p}_\diamond)^{K_\natural}$.
2. Find all the **relations** among the generators of $\mathrm{Pol}(\mathfrak{p}_\diamond)^{K_\natural}$.
3. Find an easy or effective way to express the images of the above invariant polynomials under the **Helgason map** $\Phi_{n,m}$ explicitly.
4. Decompose $\mathrm{Pol}(\mathfrak{p}_\star)^K$ into $SO(n,\mathbb{R})$**-irreducibles**.
5. Find a **complete list** of explicit generators of the algebra $\mathbb{D}(\mathfrak{P}_{n,m})$. Or construct explicit $SL_{n,m}(\mathbb{R})$-invariant differential operators on $\mathfrak{P}_{n,m}$.
6. Find all the **relations** among a set of generators of $\mathbb{D}(\mathfrak{P}_{n,m})$.
7. Is $\mathrm{Pol}(\mathfrak{p}_\diamond)^{K_\natural}$ **finitely generated**? Is $\mathbb{D}(\mathfrak{P}_{n,m})$ **finitely generated**?
8. Let $SL_{n,m}(\mathbb{Z}) := SL(n,\mathbb{Z}) \ltimes \mathbb{Z}^{(m,n)}$ denote the discrete subgroup of $SL_{n,m}(\mathbb{R})$. **Decompose** the Hilbert space $L^2(SL_{n,m}(\mathbb{Z})\backslash SL_{n,m}(\mathbb{R}))$ into **irreducible unitary representations** of $SL_{n,m}(\mathbb{R})$.

Problems in $\mathbb{D}(\mathfrak{P}_{n,m})$

Remark

Using the commutative subalgebra of $\mathbb{D}(\mathfrak{P}_{n,m})$ containing the Laplace operator, the author introduced the notion of automorphic forms for $SL_{n,m}(\mathbb{Z})$.

Remark

For the case $n = 2$, the author introduced the notion of Maass-Jacobi forms on $\mathbb{H}_{1,m}$ and investigated unitary representations of $SL_{2,m}(\mathbb{R})$.

VII. Invariant Differential Operators on $G^J/(U(n) \times S(m,\mathbb{R}))$

$G^J/(U(n)\times S(m,\mathbb{R}))$

- For two positive integers m and n, we consider the **Heisenberg group**

  $$H_{\mathbb{R}}^{(n,m)} = \{(\lambda,\mu;\kappa)\,|\,\lambda,\mu\in\mathbb{R}^{(m,n)},\ \kappa\in\mathbb{R}^{(m,m)},\ \kappa+\mu\,{}^t\lambda \text{ symmetric}\}$$

  endowed with the following multiplication

  $$(\lambda,\mu;\kappa)\circ(\lambda',\mu';\kappa') = (\lambda+\lambda',\mu+\mu';\kappa+\kappa'+\lambda\,{}^t\mu'-\mu\,{}^t\lambda')$$

  with $(\lambda,\mu;\kappa),\ (\lambda',\mu';\kappa')\in H_{\mathbb{R}}^{(n,m)}$.
- We define the **Jacobi group** G^J of degree n and index m that is the semidirect product of $Sp(2n,\mathbb{R})$ and $H_{\mathbb{R}}^{(n,m)}$

  $$G^J = Sp(2n,\mathbb{R})\ltimes H_{\mathbb{R}}^{(n,m)}$$

  endowed with the following multiplication law

  $$\begin{aligned}&\big(M,(\lambda,\mu;\kappa)\big)\cdot\big(M',(\lambda',\mu';\kappa')\big)\\ &\quad= \big(MM',(\tilde{\lambda}+\lambda',\tilde{\mu}+\mu' \quad \kappa+\kappa'+\tilde{\lambda}\,{}^t\mu'-\tilde{\mu}\,{}^t\lambda')\big) \qquad (7.1)\end{aligned}$$

  with $M,M'\in Sp(2n,\mathbb{R}),(\lambda,\mu;\kappa),\ (\lambda',\mu';\kappa')\in H_{\mathbb{R}}^{(n,m)}$ and $(\tilde{\lambda},\tilde{\mu}) = (\lambda,\mu)M'$.

$G^J/(U(n) \times S(m,\mathbb{R}))$

- **G^J acts on $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$ transitively** by

$$\big(M,(\lambda,\mu;\kappa)\big)\cdot(\Omega,Z) = \Big(M\cdot\Omega,(Z+\lambda\Omega+\mu)(C\Omega+D)^{-1}\Big), \quad (7.2)$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2n,\mathbb{R}),\ (\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)}$ and $(\Omega,Z) \in \mathbb{H}_n \times \mathbb{C}^{(m,n)}$.

- The **stabilizer K^J** of G^J at $(iI_n,0)$ is given by

$$K^J = \Big\{ \big(k,(0,0;\kappa)\big) \,\Big|\, k \in K,\ \kappa = {}^t\kappa \in \mathbb{R}^{(m,m)} \Big\},$$

where

$$K = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \,\middle|\, A\,{}^tA + B\,{}^tB = I_n,\ A\,{}^tB = B\,{}^tA,\ A,B \in \mathbb{R}^{(n,n)} \right\}.$$

- The Jacobi group G^J **is not a reductive Lie group** and the homogeneous space $\mathbb{H}_n \times \mathbb{C}^{(m,n)} \cong G^J/K^J$ is **not a symmetric space**. The homogeneous space $\mathbb{H}_{n,m} := \mathbb{H}_n \times \mathbb{C}^{(m,n)}$ is called the **Siegel-Jacobi space** of degree n and index m.

G^J-invariant metric on $\mathbb{H}_{n,m}$

Theorem 5 (J.-H. Yang, 2007)

For any two positive real numbers A and B,

$$ds^2_{n,m;A,B} = A\cdot \mathrm{Tr}\Big(Y^{-1}d\Omega\, Y^{-1}d\overline{\Omega}\Big) + B\,\Big\{\mathrm{Tr}\Big(Y^{-1}\,{}^tV\,V\,Y^{-1}d\Omega\,Y^{-1}d\overline{\Omega}\Big) + \mathrm{Tr}\Big(Y^{-1}\,{}^t(dZ)\,d\overline{Z}\Big) - \mathrm{Tr}\Big(V\,Y^{-1}d\Omega\,Y^{-1}\,{}^t(d\overline{Z})\Big) - \mathrm{Tr}\Big(V\,Y^{-1}d\overline{\Omega}\,Y^{-1}\,{}^t(dZ)\,\Big)\Big\}$$

*is a **Riemannian metric on $\mathbb{H}_{n,m}$ which is invariant under the action** (7.2) of G^J.*

Remark

J. Yang and L. Yin (2016) showed that the invariant metric $ds^2_{n,m;A,B}$ is a **Kähler metric**.

Laplace operator of $ds^2_{n,m;A,B}$

Theorem 6 (J.-H. Yang, 2007)

The ***Laplace operator*** $\Delta_{m,m;A,B}$ *of the* G^J*-invariant metric* $ds^2_{n,m;A,B}$ *is given by*

$$\Delta_{n,m;A,B} = \frac{4}{A}\,\mathbb{M}_1 + \frac{4}{B}\mathbb{M}_2, \tag{7.3}$$

where

$$\begin{aligned}\mathbb{M}_1 \;&=\; \mathrm{Tr}\left(Y\, {}^t\!\left(Y\frac{\partial}{\partial\overline{\Omega}}\right)\frac{\partial}{\partial\Omega}\right) + \mathrm{Tr}\left(VY^{-1}\,{}^tV\,{}^t\!\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial Z}\right)\\ &\quad + \mathrm{Tr}\left(V\,{}^t\!\left(Y\frac{\partial}{\partial\overline{\Omega}}\right)\frac{\partial}{\partial Z}\right) + \mathrm{Tr}\left({}^tV\,{}^t\!\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial\Omega}\right)\end{aligned}$$

and

$$\mathbb{M}_2 = \mathrm{Tr}\left(Y\,\frac{\partial}{\partial Z}\,{}^t\!\left(\frac{\partial}{\partial\overline{Z}}\right)\right).$$

Furthermore $\mathbb{M}_1$ *and* $\mathbb{M}_2$ *are* ***differential operators on*** $\mathbb{H}_{n,m}$ ***invariant under the action*** (7.2) *of* G^J.

Laplace operator of $ds^2_{n,m;A,B}$

Remark

Erik Balslev (2012) developed the spectral theory of $\Delta_{1,1;1,1}$ on $\mathbb{H}_{1,1}$ for certain arithmetic subgroups of the Jacobi modular group to prove that the set of all eigenvalues of $\Delta_{1,1;1,1}$ satisfies the Weyl law.

Remark

Yang et al (2013) proved that the **scalar curvature** of $(\mathbb{H}_{1,1}, ds^2_{1,1;A,B})$ is $-\frac{3}{A}$ and hence is independent of the parameter B. The scalar and Ricci curvatures of $(\mathbb{H}_{1,m}, ds^2_{1,m;A,B})\,(m \geq 1)$ were completely computed by G. Khan and J. Zhang (2022). Furthermore Khan and Zhang proved that $(\mathbb{H}_{1,m}, ds^2_{1,m;A,B})\,(m \geq 1)$ has non-negative orthogonal anti-bisectional curvature.

Invariant Differential Operators on $G^J/(U(n) \times S(m,\mathbb{R}))$

The **Lie algebra** $\mathfrak{g}^J$ of G^J has a **decomposition** $\mathfrak{g}^J = \mathfrak{k}^J + \mathfrak{p}^J$, where

$$\mathfrak{g}^J = \left\{ \big(Z,(P,Q,R)\big) \,\Big|\, Z \in \mathfrak{g},\ P,Q \in \mathbb{R}^{(m,n)},\ R = {}^tR \in \mathbb{R}^{(m,m)} \right\},$$

$$\mathfrak{k}^J = \left\{ \big(X,(0,0,R)\big) \,\Big|\, X \in \mathfrak{k},\ R = {}^tR \in \mathbb{R}^{(m,m)} \right\},$$

$$\mathfrak{p}^J = \left\{ \big(Y,(P,Q,0)\big) \,\Big|\, Y \in \mathfrak{m},\ P,Q \in \mathbb{R}^{(m,n)} \right\}.$$

Here

$$\mathfrak{g} = \left\{ \begin{pmatrix} X_1 & X_2 \\ X_3 & -\,{}^tX_1 \end{pmatrix} \,\Big|\, X_1, X_2, X_3 \in \mathbb{R}^{(n,n)},\ X_2 = {}^tX_2,\ X_3 = {}^tX_3 \right\},$$

$$\mathfrak{k} = \left\{ \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in \mathbb{R}^{(2n,2n)} \,\Big|\, {}^tX + X = 0,\ Y = {}^tY \right\}, \text{ and}$$

$$\mathfrak{m} = \left\{ \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \,\Big|\, X = {}^tX,\ Y = {}^tY,\ X, Y \in \mathbb{R}^{(n,n)} \right\}.$$

$\mathfrak{g}$ is the Lie algebra of $Sp(2n,\mathbb{R})$ and $\mathfrak{k}$ is the Lie algebra of $K \cong U(n)$.

- Let T_n be the vector space of all $n \times n$ **symmetric** complex matrices. For brevity, we put $T_{n,m} := T_n \times \mathbb{C}^{(m,n)}$. We define the real linear **isomorphism** $\Phi : \mathfrak{p}^J \longrightarrow T_{n,m}$ by

$$\Phi\left(\begin{pmatrix} X & Y \\ Y & -X \end{pmatrix}, (P,Q,0)\right) = \big(X + i\,Y,\, P + i\,Q\big), \tag{7.4}$$

where $\begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \in \mathfrak{m}$ and $P, Q \in \mathbb{R}^{(m,n)}$. Identifying $\mathbb{R}^{(m,n)} \times \mathbb{R}^{(m,n)}$ with $\mathbb{C}^{(m,n)}$, we can **identify $\mathfrak{p}^J$ with $T_{n,m} = T_n \times \mathbb{C}^{(m,n)}$**.

- Let $S(m,\mathbb{R})$ denote the additive group consisting of all $m \times m$ real **symmetric** matrices. Now we define the **isomorphism** $\theta : K^J \longrightarrow U(n) \times S(m,\mathbb{R})$ by

$$\theta(h,(0,0,\kappa)) = (\delta(h),\kappa), \quad h \in K,\ \kappa \in S(m,\mathbb{R}), \tag{7.5}$$

where $\delta : K \longrightarrow U(n)$ is the naturally defined group isomorphism.

- $U(n)\times S(m,\mathbb{R})$ **acts on** $T_{n,m}$ defined by

$$(h,\kappa)\cdot(\omega,z) := (h\,\omega\,{}^t h,\; z\,{}^t h), \tag{7.6}$$

where $h\in U(n),\ \kappa\in S(m,\mathbb{R}),\ (\omega,z)\in T_{n,m}$.

- If $k^J\in K^J$ and $\alpha\in\mathfrak{p}^J$, then we have the following equality

$$\Phi\Big(\mathrm{Ad}\big(k^J\big)\alpha\Big)=\theta\big(k^J\big)\cdot\Phi(\alpha). \tag{7.7}$$

- The action (7.6) induces the action of $U(n)$ on the polynomial algebra $\mathrm{Pol}_{n,m} := \mathrm{Pol}\,(T_{n,m})$. We denote by $\mathbf{Pol}_{n,m}^{U(n)}$ the subalgebra of $\mathrm{Pol}_{n,m}$ consisting of all **$U(n)$-invariants**.
- Similarly, the map (7.5) of K induces the action of K on the polynomial algebra $\mathbf{Pol}\big(\mathfrak{p}^J\big)$. We see that through the identification of $\mathfrak{p}^J$ with $T_{n,m}$, the algebra $\mathrm{Pol}\big(\mathfrak{p}^J\big)$ is **isomorphic to** $\mathrm{Pol}_{n,m}$.

Problems in $\mathbb{D}(\mathbb{H}_{n,m})$

Let $\mathbb{D}(\mathbb{H}_{n,m})$ be the algebra of **all differential operators on $\mathbb{H}_{n,m}$ that are invariant under the action** (7.2) of G^J. There is the natural linear bijection

$$\Theta_{n,m} : \mathrm{Pol}_{n,m}^{U(n)} \longrightarrow \mathbb{D}(\mathbb{H}_{n,m}).$$

1. Find a **complete list** of explicit generators of $\mathrm{Pol}_{n,m}^{U(n)}$.
2. Find all the **relations** among a set of generators of $\mathrm{Pol}_{n,m}^{U(n)}$.
3. Find an easy or effective way to express the **images of** the above invariant polynomials or generators of $\mathrm{Pol}_{n,m}^{U(n)}$ under the **Helgason map** $\Theta_{n,m}$ explicitly.
4. Decompose $\mathrm{Pol}_{n,m}$ into $U(n)$**-irreducibles**.
5. Find a **complete list** of explicit generators of the algebra $\mathbb{D}(\mathbb{H}_{n,m})$. Or construct explicit G^J**-invariant differential operators** on $\mathbb{H}_{n,m}$.
6. Find all the **relations** among a set of generators of $\mathbb{D}(\mathbb{H}_{n,m})$.
7. Is $\mathrm{Pol}_{n,m}^{U(n)}$ **finitely generated**? Is $\mathbb{D}(\mathbb{H}_{n,m})$ **finitely generated**?
8. Are there **canonical ways to find** generators of $\mathrm{Pol}_{n,m}^{U(n)}$?

Problems in $\mathbb{D}(\mathbb{H}_{n,m})$

Theorem 7 (Minoru Itoh, 2013)

We put $\varphi^{(2k)} = \mathrm{Tr}((w\bar{w})^k)$*. Moreover, for* $1 \leq a, b \leq m$ *and* $k \geq 0$*, we put*

$$\psi_{ba}^{(0,2k,0)} = (\bar{z}(w\bar{w})^k \, {}^t z)_{ba}, \qquad \psi_{ba}^{(1,2k,0)} = (z\bar{w}(w\bar{w})^k \, {}^t z)_{ba}.$$

The algebra $\mathrm{Pol}_{n,m}^{U(n)}$ ***is generated by*** *the following polynomials:*

$$\varphi^{(2k+2)}, \, \mathrm{Re}\, \psi_{ab}^{(0,2k,0)}, \, \mathrm{Im}\, \psi_{cd}^{(0,2k,0)}, \, \mathrm{Re}\, \psi_{ab}^{(1,2k,0)}, \, \mathrm{Im}\, \psi_{ab}^{(1,2k,0)}.$$

Here the indices run as follows:

$$0 \leq k \leq n-1, \quad 1 \leq a \leq b \leq m, \quad 1 \leq c < d \leq m.$$

Problems in $\mathbb{D}(\mathbb{H}_{1,1})$

For **the case when** $n = m = 1$, the above eight problems are completely solved.

Theorem 8 (J.-H. Yang, 2003)

For a coordinate (w, ξ) *in* $T_{1,1} = \mathbb{C} \times \mathbb{C}$, *we write* $w = r + i\,s$, $\xi = \zeta + i\,\eta \in \mathbb{C}$, r, s, ζ, η *real. The algebra* $\mathrm{Pol}_{1,1}^{U(1)}$ ***is generated*** *by*

$$q(w,\xi) = \frac{1}{4}\,w\,\overline{w} = \frac{1}{4}\big(r^2 + s^2\big), \qquad \alpha(w,\xi) = \xi\,\overline{\xi} = \zeta^2 + \eta^2,$$

$$\phi(w,\xi) = \frac{1}{2}\,\mathit{Re}\,\big(\xi^2\overline{w}\big) = \frac{1}{2}\,r\big(\zeta^2 - \eta^2\big) + s\,\zeta\eta,$$

$$\psi(w,\xi) = \frac{1}{2}\,\mathit{Im}\,\big(\xi^2\overline{w}\big) = \frac{1}{2}\,s\big(\eta^2 - \zeta^2\big) + r\,\zeta\eta.$$

Problems in $\mathbb{D}(\mathbb{H}_{1,1})$

Theorem 9 (J.-H. Yang, 2003)

Let $D_1 = \Theta_{1,1}(q)$, $D_2 = \Theta_{1,1}(\alpha)$, $D_3 = \Theta_{1,1}(\phi)$ ***and*** $D_4 = \Theta_{1,1}(\psi)$. *Then,*

$$D_1 = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + v^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + 2\, y\, v \left(\frac{\partial^2}{\partial x \partial u} + \frac{\partial^2}{\partial y \partial v} \right),$$

$$D_2 = y \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right),$$

$$D_3 = y^2 \frac{\partial}{\partial y} \left(\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} \right) - 2y^2 \frac{\partial^3}{\partial x \partial u \partial v} - \left(v \frac{\partial}{\partial v} + 1 \right) D_2,$$

and

$$D_4 = y^2 \frac{\partial}{\partial x} \left(\frac{\partial^2}{\partial v^2} - \frac{\partial^2}{\partial u^2} \right) - 2y^2 \frac{\partial^3}{\partial y \partial u \partial v} - v \frac{\partial}{\partial u} D_2,$$

where $\tau = x + iy$ *and* $z = u + iv$ *with real variables* x, y, u, v.

Problems in $\mathbb{D}(\mathbb{H}_{1,1})$

Remark

By Theorem 9, the following relation holds.

$$D_1D_2 - D_2D_1 = 2\,y^2\,\frac{\partial}{\partial y}\left(\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2}\right) - 4\,y^2\,\frac{\partial^3}{\partial x \partial u \partial v} - 2\left(v\,\frac{\partial}{\partial v}D_2 + D_2\right).$$

Therefore, the algebra $\mathbb{D}(\mathbb{H}_{1,1})$ **is not commutative**.

Lemma 10 (Hiroyuki Ochiai, 2013)

We have the following relation

$$\phi^2 + \psi^2 = q\,\alpha^2.$$

This relation exhausts all the relations *among the generators q, α, ϕ and ψ of $\mathrm{Pol}_{1,1}^{U(1)}$.*

Problems in $\mathbb{D}(\mathbb{H}_{1,1})$

Theorem 11 (Hiroyuki Ochiai, 2013)

We have the following relations

(a) $[D_1, D_2] = 2D_3$.

(b) $[D_1, D_3] = 2D_1D_2 - 2D_3$.

(c) $[D_2, D_3] = -D_2^2$.

(d) $[D_4, D_1] = 0$.

(e) $[D_4, D_2] = 0$.

(f) $[D_4, D_3] = 0$.

(g) $D_3^2 + D_4^2 = D_2D_1D_2$.

*These **seven relations exhaust all the relations** among the generators D_1, D_2, D_3 and D_4 of $\mathbb{D}(\mathbb{H}_{1,1})$.*

David Hilbert (1862 -1943)
Hermann Weyl (1885 -1955)
Sigurdur Helgason (1927 - 2023)
Bertram Kostant (1928 - 2017)
Raoul Bott (1923 - 2005)
Roger E. Howe (1945 -)
Capelli's Identity, Reductive Dual Pair
Alfredo Capelli (1855 -1910)

Thank you!