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- Explicit construction of mock modular forms

♣ Let

- $Q = [a, b, c] \in C(D)$. we can choose a matrix M_Q such that

$$M_Q \equiv \begin{cases} \begin{pmatrix} a & \frac{b}{2} \\ 0 & 1 \end{pmatrix} & \text{if } p \nmid a \\ \begin{pmatrix} -\frac{b}{2} & -c \\ 1 & 0 \end{pmatrix} & \text{if } p \mid a \text{ and } p \nmid c \pmod{p^{r_p}} \text{ for } D \equiv 0 \pmod{4} \\ \begin{pmatrix} -\frac{b}{2} - a & -\frac{b}{2} - c \\ 1 & -1 \end{pmatrix} & \text{if } p \mid a \text{ and } p \mid c \end{cases}$$

and

$$M_Q \equiv \begin{cases} \begin{pmatrix} a & \frac{b-1}{2} \\ 0 & 1 \end{pmatrix} & \text{if } p \nmid a \\ \begin{pmatrix} -\frac{b-1}{2} & -c \\ 1 & 0 \end{pmatrix} & \text{if } p \mid a \text{ and } p \nmid c \pmod{p^{r_p}} \text{ for } D \equiv 3 \pmod{4} \\ \begin{pmatrix} -\frac{b-1}{2} - a & \frac{1-b}{2} - c \\ 1 & -1 \end{pmatrix} & \text{if } p \mid a \text{ and } p \mid c \end{cases}$$

where p runs over all prime factors of N and $p^{r_p} \parallel N$.

Setting

♣ Let

- $L \subset V(\mathbb{Q})$: an even lattice of full rank
- $L^\#$: the dual lattice of L
- Γ : a congruence subgroup preserves L and acts trivially on $L^\#/L$

♣ For $X \in V(\mathbb{Q})$ of positive norm we put

$$D_X = \text{span}(X) \in D.$$

For $m \in \mathbb{Q}_{>0}$ and $h \in L^\#$, the group Γ acts on

$$L_{h,m} = \{X \in L + h \mid q(X) = m\}$$

with finitely many orbits.

- ♣ Counting the number of possible β , we get

$$\mathrm{GT}_{\zeta_3^B \gamma_2}(D) = \frac{1}{4} \mathrm{MT}_{\gamma_2}^{L_1}(0, D).$$

- ♣ Using the above relation, Jeon, Kang and Kim proved that

$$q^{-1} + \sum_{\substack{D>0, (3, D)=1, \\ -D \equiv \square \pmod{36}}} \mathrm{GT}_{\zeta_3^B \gamma_2}(D) q^D$$

is a weakly holomorphic modular form of weight $3/2$ on $\Gamma_0(36)$.

- ♣ This method can be applied to other ring class invariants or ray class invariants.

Let p be one or a prime and $\Gamma_0^+(p)$ be the group generated by the Hecke group $\Gamma_0(p)$ and the Fricke involution $W_p = \begin{pmatrix} 0 & -1/\sqrt{p} \\ \sqrt{p} & 0 \end{pmatrix}$.

Definition

Let k be a positive integer greater than 2 and let P_{k-2} denote the space of all polynomials of degree at most $k-2$. For $p \in \{1, 2, 3\}$ we define a subspace W_{k-2}^+ of P_{k-2} by

$$\begin{aligned} W_{k-2}^+ &= \{g \in P_{k-2} \mid g + g|_{2-k}W_p = 0 \\ &= g + g|_{2-k}U + g|_{2-k}U^2 + \cdots + g|_{2-k}U^{n_p-1}\} \end{aligned}$$

with $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $U = TW_p$, and $n_p = \begin{cases} 3, & \text{if } p = 1 \\ 2p, & \text{if } p = 2, 3 \end{cases}$. The elements of the space W_{k-2}^+ are called *period polynomials*.

Period Polynomials

- When $p = 1, 2, 3$, period polynomials have been investigated in relation to modular integrals, cusp forms via the Eichler-Shimura isomorphism and to various other areas of mathematics.
- The importance of period polynomials comes from their close connection with special values of modular L -functions.

Basis for $M_k^{1,\epsilon}(p)$

- When the genus of $\Gamma_0^+(p)$ is zero, the space $M_k^{1,\epsilon}(p)$ has a canonical basis.
- When $p = 1$, such a canonical basis was constructed by Duke and Jenkins.
- Let m_k^ϵ denote the maximal order of a nonzero $f \in M_k^{1,\epsilon}(p)$ at ∞ . Indeed, for every integer $m \geq -m_k^\epsilon$, there exists a unique weakly holomorphic modular form $f_{k,m}^\epsilon \in M_k^{1,\epsilon}(p)$ with Fourier expansion of the form

$$f_{k,m}^\epsilon(\tau) = q^{-m} + \sum_{n > m_k^\epsilon} a_k^\epsilon(m, n) q^n$$

and together they form a basis for $M_k^{1,\epsilon}(p)$.

- If $k > 2$, the maximal order m_k^ϵ is given by $\dim S_k^\epsilon(p)$ where $S_k^\epsilon(p)$ denotes the space of holomorphic cusp forms f of weight k for $\Gamma_0(p)$ with $f|_k W_p = \epsilon f$.

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Eichler integrals and period polynomials

Definition

For $f = \sum_{n \gg -\infty} a_f(n)q^n \in M_k^{!,+}(p)$ we define the **Eichler integral of f** by

$$\mathcal{E}_f(z) := \sum_{\substack{n \gg -\infty \\ n \neq 0}} a_f(n)n^{1-k}q^n.$$

Definition

The **period polynomial for f** is defined by

$$r^+(f)(z) := c_k(\mathcal{E}_f - \mathcal{E}_f|_{2-k}W_p)(z) \quad \text{with} \quad c_k = -\frac{\Gamma(k-1)}{(2\pi i)^{k-1}},$$

which measures the obstruction to modularity of the Eichler integral of f .

- Let $S_k^{!,+}(p)$ be the subspace of $M_k^{!,+}(p)$ consisting of weakly holomorphic modular forms for $\Gamma_0^+(p)$ with zero constant term in the Fourier expansion.
- Bol's identity** states that for any $\gamma \in SL_2(\mathbb{R})$ and any function g , we have

$$(1) \quad \frac{d^{k-1}}{dz^{k-1}}(g|_{2-k}\gamma) = \left(\frac{d^{k-1}g}{dz^{k-1}}\right)|_k\gamma,$$

from which it easily follows that $r^+(f)(z) \in P_{k-2}$ for each $f \in S_k^{!,+}(p)$.

- Moreover r^+ defines a map from $S_k^{!,+}(p)$ to W_{k-2}^+ .

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Thus we have

$$\dim S_k^{!,+}(p)/D^{k-1}(M_{2-k}^{!,+}(p)) = 2 \dim S_k^+(p).$$

Question: Is there a nice basis for $S_k^{!,+}(p)/D^{k-1}(M_{2-k}^{!,+}(p))$ consisting of Hecke eigenforms?

Multiplicity Two Theorem

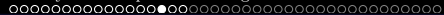
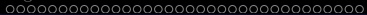
If $t = \dim S_k^+(p)$, then $\dim W_{k-2}^+ = 2t + 1$ and $\dim S_k^{!,+}(p)/D^{k-1}(M_{2-k}^{!,+}(p)) = 2t$.

Theorem (2013, Bringmann, Guerzhoy, Kent and Ono)

When $p = 1$,

$$S_k^{!,+}(p)/D^{k-1}(M_{2-k}^{!,+}(p)) = \bigoplus \mathbb{T}_i$$

where each \mathbb{T}_i consists of a cuspidal Hecke eigenform and a weakly holomorphic Hecke eigenform with the same eigenvalues.

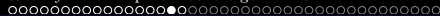
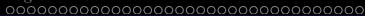


Quotient Space $\widehat{S}_k^+(p)$

- Following Guerzhoy, we set

$$\widehat{S}_k^+(p) := \frac{S_k^{!,+}(p)}{D^{k-1}(M_{2-k}^{!,+}(p)) \oplus S_k^+(p)}.$$

- If $t = \dim S_k^+(p)$, then $\dim \widehat{S}_k^+(p) = t$ and $\widehat{S}_k^+(p) = \langle \widehat{f}_{k,1}, \dots, \widehat{f}_{k,t} \rangle$.
- The pairing $\{\cdot, \cdot\}$ induces a non-degenerate pairing on $\widehat{S}_k^+(p) \times S_k^+(p)$.



Dual Basis

- Let $t = \dim S_k^+(p)$ and

$$\{f_n = \sum_{m>0} \lambda(n, m) q^m \mid n = 1, 2, \dots, t\}$$

be a basis of $S_k^+(p)$ consisting of normalized Hecke eigenforms.

- Let

$$f_n^* = \sum_{m \geq -t} \mu(m, n) q^m$$

with $\mu(m, n) \in \mathbb{C}$ be a linear combination of $f_{k,1}, \dots, f_{k,t}$ which is dual to f_n with respect to the pairing, i.e. $\{f_m^*, f_n\} = \delta_{mn}$ where δ_{mn} is the Kronecker delta function. Then such functions f_n^* are unique.

Dual Basis

Moreover $T_l f_n^*$ and $\lambda(n, l) f_n^*$ represent the same coset in $S_k^{!,+}(p)/(D^{k-1}(M_{2-k}^{!,+}(p)) \oplus S_k^+(p))$. Thus we can write

$$(3) \quad T_l f_n^* = \lambda(n, l) f_n^* + D^{k-1} g_{n,l} + \sum_{j=1}^t a_{jn}(l) f_j$$

for some $g_{n,l} \in M_{2-k}^{!,+}(p)$ and $a_{jn}(l) \in \mathbb{C}$.

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Good Maass form

- The **existence** of \mathfrak{F} , which is good for a Hecke eigenform f^c is guaranteed by Bruinier, Ono and Rhoades' work.
- Let \mathfrak{F}_0 be good for a Hecke eigenform f^c and denote $t = \dim S_k^+(p)$ and $t' = \dim S_k^-(p)$. Let $M_{2-k}^\sharp(p)$ be the space of weakly holomorphic modular forms of weight $2-k$ for $\Gamma_0(p)$ with poles allowed only at the cusp ∞ . For $p \in \{1, 2, 3, 5, 7, 13\}$, it follows from Ahn and Choi's work that

$$\max\{\text{ord}_\infty f \mid f \neq 0 \in M_{2-k}^\sharp(p)\} = -1 - t - t',$$

and for each integer m with $-m \leq -1 - t - t'$, there exists

$f_{2-k,m}^\sharp = q^{-m} + O(q^{-t-t'}) \in M_{2-k}^\sharp(p)$ with integral Fourier coefficients. By subtracting a suitable linear combination of $f_{2-k,m}^\sharp$'s from \mathfrak{F}_0 we can take a **unique** \mathfrak{F} which is good for f^c and $\mathfrak{F}^+ = O(q^{-t-t'})$.

Good Maass form

- The structure of half-integral weight weakly holomorphic Hecke eigenforms was developed and half-integral weight p -adic coupling was investigated by Bringmann, Guerzhoy and Kane.

Example of good Maass form of level 1

Let $p = 1$ and $k = 12$. In this case we have $t = \dim S_{12}^+(1) = 1$. Using $\Delta(z) = \eta(z)^{24} \in S_{12}^+(1)$ and the Hauptmodul $j_1(z) = E_4(z)^3/\Delta(z) - 744$ for $\Gamma_0^+(1)$ one can express $f_{12,m}$ ($-1 \leq m \leq 1$) as follows:

$$f_{12,-1}(z) = \Delta(z) = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 + \dots$$

$$f_{12,0}(z) = \Delta(j_1 + 24) = 1 + 196560q^2 + 16773120q^3 + \dots$$

$$f_{12,1}(z) = \Delta(j_1^2 + 24j_1 - 393444) = \frac{1}{q} + 47709536q^2 + \dots$$

Example of good Maass form of level 1

$$f_1 = f_{12, -1} \quad \text{and} \quad h_1 = f_1^* = f_{12, 1}$$

$$F_\alpha = -h_1 = -f_{12, 1} = -q^{-1} - \sum_{n=2}^{\infty} a_{12}(1, n)q^n,$$

If we take $l = 3$ and $w = 1$ in (*), by using Sturm bound one verifies that

$$\begin{aligned} & \frac{-\sum_{n=1}^{\infty} a_{12}(1, 3n)q^n}{-a_{12}(1, 3)} \\ &= q + \frac{27947672851540608}{39862705122}q^2 + \frac{340389905850815087232}{39862705122}q^3 + \dots \\ &\equiv \Delta \pmod{3^{10}}. \end{aligned}$$

Example of good Maass form of level 5

$$F_\alpha = -h_1 + D^9(w_1 + \beta_{11} f_{-8,3}^-) + \frac{4}{25} f_1 = \sum_{n=-3}^{\infty} c_\alpha(n) q^n$$

Take $l = 3$ and $w = 1$ in (*). Using Sturm bound one verifies that

$$\begin{aligned} & \frac{\sum_{n=-1}^{\infty} c_\alpha(3n) q^n}{c_\alpha(3)} \\ &= -\frac{6561}{6308q} - \frac{18528264}{1577} q^2 + \frac{808269273}{1577} q^3 + \frac{68622811200}{1577} q^4 + \dots \\ &\equiv f_1 \pmod{3^8}. \end{aligned}$$

Thank you for your attention.