Arithmetic of weakly holomorphic modular forms

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    - Class fields over an imaginary quadratic field
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2. Weakly holomorphic Hecke eigenforms
    - Construction of Weakly holomorphic Hecke eigenforms in higher level cases
    - Explicit construction of mock modular forms

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j-function

♣ For a lattice $\Lambda = [\lambda_1,\ \lambda_2]$ in $\mathbb{C}$, we define the constants

$$\begin{aligned} g_2(\Lambda) &= 60 \sum_{\omega\in\Lambda\setminus\{0\}} \frac{1}{\omega^4} \\ g_3(\Lambda) &= 140 \sum_{\omega\in\Lambda\setminus\{0\}} \frac{1}{\omega^6} \\ \Delta(\Lambda) &= g_2(\Lambda)^3 - 27g_3(\Lambda)^2 \text{ (in fact, } \neq 0) \\ j(\Lambda) &= 1728\frac{g_2(\Lambda)^3}{\Delta(\Lambda)}. \end{aligned}$$

♣Two lattices Λ_1 and Λ_2 are homothetic if there is a constant $\alpha \in \mathbb{C}^*$ such that $\Lambda_2 = \alpha\Lambda_1$.

♣ It is known that

$$j(\Lambda_1) = j(\Lambda_2) \Longleftrightarrow \Lambda_1 \text{ and } \Lambda_2 \text{ are homothetic.}$$

♣ We define the elliptic modular function

$$j(\tau) = j([\tau,\ 1]) \quad (\tau \in \mathbb{H}).$$

where $[\tau,\ 1] = \mathbb{Z}\tau + \mathbb{Z}$ is a lattice.

♣ Then it is a classical modular function for $\mathrm{SL}_2(\mathbb{Z})$ and

$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots\ , \quad \text{with } q = e^{2\pi i\tau}.$$

Action on $\mathbb{H}^*$

♣ Let

$$\mathbb{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im}(\tau) > 0\}$$

be the complex upper half plane and

$$\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$$

be the extended upper half plane.

♣The modular group

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ : \ a, b, c, d \in \mathbb{Z} \ \text{ with } \ ad - bc = 1 \right\}$$

acts on $\mathbb{H}^*$ by linear fractional transformation

$$\tau \ \mapsto \ \gamma\tau = \frac{a\tau + b}{c\tau + d}$$

for $\tau = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$.

Two elements of $\mathrm{SL}_2(\mathbb{Z})$ give the same action on $\mathbb{H}^*$ $\iff$ they differ by $\pm\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Congruence subgroups

♣ For a positive integer N, let

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \ : \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$$

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \ : \ c \equiv 0 \pmod{N} \right\}$$

$$\Gamma_0^0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \ : \ b \equiv c \equiv 0 \pmod{N} \right\}.$$

♣ $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$.

♣ If $\Gamma(N) \subset \Gamma \subset \Gamma(1)$, then Γ is called a congruence subgroup.

Modular functions

♣ Let Γ be a congruence subgroup. We say that a meromorphic function f on $\mathbb{H}$ is a modular function for Γ if

(i) $f(\gamma\tau) = f(\tau)$ for all $\gamma \in \Gamma$,

(ii) $f(\tau)$ is meromorphic at every cusp.

(ii) means that for each $\gamma \in SL_2(\mathbb{Z})$, $f \circ \gamma$ has the Fourier expansion

$$f \circ \gamma(\tau) = \sum_{n \geq m}^{\infty} c_n \underbrace{\left(q^{\frac{1}{N}}\right)}_{\text{parameter}}{}^{n} \quad (c_n \in \mathbb{C}, \quad m \in \mathbb{Z}).$$

♣ $j(\tau)$ is a modular function for $\Gamma(1)$.

♣ A modular function for $\Gamma(N)$ is called a modular function of level N.

♣ We denote

$$\mathbb{C}(X(N)) = \text{the field of all modular functions of level } N.$$

In particular, $\mathbb{C}(X(1)) = \mathbb{C}(j(\tau))$.

♣ $\mathbb{C}(X(N))$ is a Galois extension of $\mathbb{C}(X(1))$. Its Galois group is given by

$$\Gamma(1)/\pm\Gamma(N) \simeq \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1_2\}$$

whose action is composition as linear fractional transformation.

♣ For a positive integer N, let

$$\mathcal{F}_N = \text{the field of functions in } \mathbb{C}(X(N)) \text{ whose Fourier expansion have coefficients in } \mathbb{Q}(\zeta_N), \text{ where } \zeta_N = e^{\frac{2\pi i}{N}}.$$

♣ $\mathcal{F}_1 = \mathbb{Q}(j(\tau))$.

Galois group $\mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_1)$

♣ $\mathcal{F}_N$ is a Galois extension of $\mathcal{F}_1$ and its Galois group is represented by

$$\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1_2\} = G_N \times \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1_2\}$$

where

$$G_N = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \;\middle|\; d \in (\mathbb{Z}/N\mathbb{Z})^* \right\}.$$

(i) The matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}\right) \in G_N$ acts on $\mathcal{F}_N$ by

$$\sum_{n \geq m}^{\infty} c_n (q^{\frac{1}{N}})^n \mapsto \sum_{n \geq m}^{\infty} c_n^{\sigma_d} (q^{\frac{1}{N}})^n$$

where σ_d is the automorphism of $\mathbb{Q}(\zeta_N)$ induced by $\zeta_N \mapsto \zeta_N^d$.

(ii) An element $\gamma \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1_2\}$ acts on $\mathcal{F}_N$ by composition as linear transformation.

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Orders

♣ Let

- K : an imaginary quadratic field with discriminant d_K
- D : a positive integer $\equiv 0,\ 3 \pmod 4$ s.t. $D = -f^2 \cdot d_K$

♣ Let

$$\tau_D = \begin{cases} \dfrac{\sqrt{-D}}{2} & \text{for } D \equiv 0 \pmod 4 \\ \dfrac{-1+\sqrt{-D}}{2} & \text{for } D \equiv 3 \pmod 4 \end{cases}$$

♣ $\mathcal{O}_D = [1,\ \tau_D]$ is an order of conductor f with discriminant $-D$ in K.

♣ When $f = 1$, $\mathcal{O}_D = \mathcal{O}_K$ is the ring of integers of K, which is the maximal order in K.

Ring class field of order $\mathcal{O}$

♣ Let

- K : an imaginary quadratic field
- $\mathcal{O}$: an order of conductor f with discriminant $-D$ in K

♣ The quotient group

$$\mathrm{Cl}(\mathcal{O}) := \frac{\left\{ \text{fractonal ideals of } K \text{ prime to } f \right\}}{\left\langle (\alpha) \mid \alpha \in \mathcal{O}_K,\ \alpha \equiv a \pmod{f},\ a \in \mathbb{Z},\ (a,\ f) = 1 \right\rangle}$$

is called the ring class group of order $\mathcal{O}$.

♣ By the existence theorem of class field theory, there exists unique abelian extension $H_{\mathcal{O}}$ of K such that

$$\mathrm{Gal}(H_{\mathcal{O}}/K) \cong \mathrm{Cl}(\mathcal{O}),$$

which is called ring class field of order $\mathcal{O}$.

Extended ring class field of $\mathcal{O}$

♣ We define the quotient group

$$\mathrm{Cl}(\mathcal{O}, N) \quad := \quad \frac{\left\{ \text{fractonal ideals of } K \text{ prime to } f \cdot N \right\}}{\left\langle (\alpha) \mid \alpha \in \mathcal{O}_K,\ \alpha \equiv a \pmod{f \cdot N},\ a \in \mathbb{Z},\ a \equiv 1 \pmod{N} \right\rangle}.$$

♣ We can consider the extension $H_{\mathcal{O},N}$ of K with Galois group

$$\mathrm{Gal}(H_{\mathcal{O},N}/K) \ \cong\ \mathrm{Cl}(\mathcal{O}, N).$$

♣ We call $H_{\mathcal{O},N}$ the extended ring class field of $\mathcal{O}$ of level N.

♣ In particular,

$$\begin{aligned} H_{\mathcal{O}_K} \quad &= \quad \text{the Hilbert class field of } K \text{ and} \\ H_{\mathcal{O}_K,N} \quad &= \quad \text{the ray class field of modulus } N\mathcal{O}_K \text{ of } K. \end{aligned}$$

Outline

Theory of complex multiplication

♣ Let

- K : an imaginary quadratic field of discriminant d_K
- $\mathcal{O} = [1,\ \tau_D] \subset \mathcal{O}_K$: an order of conductor f and discriminant $-D$

♣ From the theory of complex multiplication, we have

$$\begin{aligned} H_{\mathcal{O}} &= K\Big(j(\mathcal{O})\Big) = K\Big(j(\tau_D)\Big), \\ H_{\mathcal{O},N} &= K\Big(f(\tau_D) \mid f \in \mathcal{F}_N \text{ is finite at } \tau_D\Big). \end{aligned}$$

Shimura's reciprocity law

♣ Assume that $K \neq \mathbb{Q}(\sqrt{-1}),\ \mathbb{Q}(\sqrt{-3})$.

♣ Let $\min(\tau_D,\ \mathbb{Q}) = X^2 + Bx + C \in \mathbb{Z}[X]$, and let

$$W_{\mathcal{O},\ N} = \left\{ \begin{bmatrix} t - Bs & -Cs \\ s & t \end{bmatrix} \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \mid t,\ s \in (\mathbb{Z}/N\mathbb{Z}) \right\}$$

which is a subgroup of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$.

♣ Then, there is an isomorphism

$$\begin{aligned} W_{\mathcal{O},\ N}/\{\pm I_2\} \quad &\to \quad \mathrm{Gal}(H_{\mathcal{O},N}/H_{\mathcal{O}}) \\ \gamma \quad &\mapsto \quad \Big(f(\tau_D) \mapsto f^{\gamma}(\tau_D) \mid f \in \mathcal{F}_N \text{ is finite at } \tau_D \Big). \end{aligned}$$

G. Shimura,
Introduction to the Arithmetic Theory of Automorphic Functions,
Iwanami Shoten and Princeton Univ. Press, 1971.

P. Stevenhagen,
Hilbert's 12th problem, complex multiplication and Shimura reciprocity,
Adv. Stud. Pure Math. 30, 2001.

Form class group

♣ Let

- $\mathcal{Q}_D$: the set of positive definite binary quadratic forms

$$Q = ax^2 + bxy + cy^2 \quad \in \mathbb{Z}[x,\ y]$$

with discriminant $b^2 - 4ac = -D$.

- Q_D^0 : the subset of $\mathcal{Q}_D$ of primitive forms. (i.e. Q with $(a,\ b,\ c) = 1$).

♣ Then $\overline{\Gamma(1)} = \mathrm{SL}_2(\mathbb{Z})/\pm I_2$ acts on $\mathcal{Q}_D$ by

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \ : \ Q(x,y) \ \mapsto \ Q(ax+by, cx+dy)$$

♣ It is well known that

$$\mathrm{Gal}(H_{\mathcal{O}}/K) \cong C(D) = Q_D^0/\overline{\Gamma(1)}.$$

♣ Let

- $Q = [a,\ b,\ c] \in \mathrm{C}(D)$. we can choose a matrix M_Q such that

$$M_Q \equiv \begin{cases} \begin{pmatrix} a & \frac{b}{2} \\ 0 & 1 \end{pmatrix} & \text{if } p \nmid a \\ \begin{pmatrix} -\frac{b}{2} & -c \\ 1 & 0 \end{pmatrix} & \text{if } p \mid a \text{ and } p \nmid c \pmod{p^{r_p}} \text{ for } D \equiv 0 \pmod 4 \\ \begin{pmatrix} -\frac{b}{2} - a & -\frac{b}{2} - c \\ 1 & -1 \end{pmatrix} & \text{if } p \mid a \text{ and } p \mid c \end{cases}$$

and

$$M_Q \equiv \begin{cases} \begin{pmatrix} a & \frac{b-1}{2} \\ 0 & 1 \end{pmatrix} & \text{if } p \nmid a \\ \begin{pmatrix} \frac{-b-1}{2} & -c \\ 1 & 0 \end{pmatrix} & \text{if } p \mid a \text{ and } p \nmid c \pmod{p^{r_p}} \text{ for } D \equiv 3 \pmod 4 \\ \begin{pmatrix} \frac{-b-1}{2} - a & \frac{1-b}{2} - c \\ 1 & -1 \end{pmatrix} & \text{if } p \mid a \text{ and } p \mid c \end{cases}$$

where p runs over all prime factors of N and $p^{r_p} || N$.

♣ For $Q = [a,\ b,\ c] \in \mathrm{C}(D)$, let

$$\tau_Q = \frac{-b + \sqrt{-D}}{2a}.$$

♣ Then, we have an injective map

$$\begin{array}{ccc} \mathrm{C}(D) & \to & \mathrm{Gal}(H_{\mathcal{O},N}/K) \\ Q^{-1} & \mapsto & \Big(f(\tau_K) \mapsto f^{M_Q}(\tau_Q) \mid f \in \mathcal{F}_N \text{ is finte at } \tau_D \Big). \end{array}$$

The restriction to $H_{\mathcal{O}}$, followed by the above map, gives rise to the isomorphism

$$\mathrm{C}(D) \ \simeq \ \mathrm{Gal}(H_{\mathcal{O}}/K).$$

P. Stevenhagen,
Hilbert's 12th problem, complex multiplication and Shimura reciprocity,
Adv. Stud. Pure Math. 30, 2001.

♣ In summary, if $N \geq 2$,

$$
\begin{array}{ccc}
 & & H_{\mathcal{O},N} \\
W_{\mathcal{O},\ N}/\{\pm I_2\} & & \Big(\ \Big| \\
 & & H_{\mathcal{O}} \ = \ K\Big(j(\tau_D)\Big) \\
\mathrm{C}(D) & & \Big(\ \Big| \\
 & & K
\end{array}
$$

Trace of singular moduli

♣ For a positive integer $D \equiv 0,\ 3 \pmod 4$, $j(\tau_D)$ generates the ring class field $H_{\mathcal{O}}$ over $K = \mathbb{Q}(\tau_D)$.

♣ The Galois conjugates of $j(\tau_D)$ under the action of $\mathrm{Gal}(H_{\mathcal{O}}/K)$ are singular moduli $j(\tau_Q)$ for $Q \in C(D)$.

♣ Let $J(\tau) = j(\tau) - 744$ be the normalized Hauptmodul for $\Gamma(1)$.

♣ D. Zagier defined the modified Galois trace

$$t_J(D) = \sum_{Q \in \mathcal{Q}_D/\overline{\Gamma(1)}} \frac{J(\tau_Q)}{|\overline{\Gamma(1)}_Q|},$$

where $\overline{\Gamma(1)}_Q$ is the stabilizer of Q

♣ Zagier showed that the generating series

$$-q^{-1} + 2 + \sum_{D=1}^{\infty} t_J(D) q^D = -q^{-1} + 2 - 248q^3 + 492q^4 - 4119q^7 + 7256q^8 + \cdots$$

is a weakly holomorphic modular form of weight 3/2 for the group $\Gamma_0(4)$.

Outline

Setting

♣ We consider the quadratic space $(V,\ q)$ given by

$$V(\mathbb{Q}) = \left\{ X = \begin{pmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{pmatrix} \in M_2(\mathbb{Q}) \right\}$$

with $q(X) = \det(X)$ and $(X, Y) = -\mathrm{tr}(XY)$. $\mathrm{SL}_2(\mathbb{Q})$ acts on V by conjugation, i.e. $\gamma.X = \gamma X \gamma^{-1}$. Let D be the space of positive lines in $V(\mathbb{R})$.

♣ Then $\mathbb{H}$ is identified with D as follows.

$$\begin{array}{rcl} \mathbb{H} & \approx & D \\ i & \mapsto & \mathrm{span}(X_0) \quad \text{with } X_0 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right) \\ z = x + iy & \mapsto & X(z) = g_z.X_0 \quad \text{with } g_z = \sqrt{y}^{-1} \left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix} \right) \end{array}$$

$\Rightarrow X(z) = \frac{1}{y} \left(\begin{smallmatrix} -x & x^2+y^2 \\ -1 & x \end{smallmatrix} \right)$

Setting

♣ Let

- $L \subset V(\mathbb{Q})$: an even lattice of full rank
- $L^{\#}$: the dual lattice of L
- Γ : a congruence subgroup preserves L and acts trivially on $L^{\#}/L$

♣ For $X \in V(\mathbb{Q})$ of positive norm we put

$$D_X = \mathrm{span}(X) \in D.$$

For $m \in \mathbb{Q}_{>0}$ and $h \in L^{\#}$, the group Γ acts on

$$L_{h,m} = \{X \in L + h \mid q(X) = m\}$$

with finitely many orbits.

Kudla-Millson Theta series

♣ For $h \in L^{\#}/L$, $\tau = u + iv$, and $z = x + iy$, consider the theta kernel

$$\theta_h(\tau, z, \varphi) = \sum_{X \in L+h} \varphi(X, \tau, z)$$

where

$$\varphi(X, \tau, z) = \left(v(X, X(z))^2 - \frac{1}{2\pi} \right) e^{2\pi i q(X)\tau - \pi v(X,X(z))^2 + 2\pi v(X,X)} \frac{dxdy}{y^2}$$
$$\in \Omega^{1,1}(D) \;\; (= \text{ the closed differential forms on } D \text{ of Hodge type } (1,1))$$

♣ Properties of the theta kernel:

- $\theta_h(\tau, z, \varphi)$ is a Γ-invariant differential form in z.
- $\theta_h(\tau, z, \varphi)$ transforms as a non-holomorphic modular form of weight $3/2$ for $\Gamma(m)$ where m is the level of the lattice L, i.e. m is the smallest positive integer N such that $Nq(x) \in \mathbb{Z}$ for all $X \in L^{\#}$.
- Let $\sigma \in \Gamma(1)$. There is a constant $C > 0$ such that

$$\theta_h(\tau, \sigma z) = O(e^{-Cy^2}), \quad y \to \infty$$

uniformly in x.

Theta integral

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Modularity of modular traces

♣ For a weakly holomorphic modular function f for Γ, Bruinier-Funke defined the modular trace of f for positive index m by

$$\mathrm{MT}_f^L(h,\ m) = \sum_{X \in \Gamma \backslash L_{h,m}} \frac{1}{|\overline{\Gamma}_X|} f(D_X).$$

♣ Assume that the constant coefficients of f at all cusps vanish. They proved that

$$\sum_{n >> -\infty} \mathrm{MT}_f^L(h,\ n) q^n$$

is a weakly holomorphic modular form of weight $3/2$ for $\Gamma(4N)$, where $4N$ is the level of the lattice L.

♣ If $h = 0$, the above series is modular for the bigger group $\Gamma_0(4N)$.

Modularity of Galois trace

♣ Let

- N : a positive integer
- D : a positive integer such that $-D \equiv \square \pmod{4N^2}$
- $\beta \in \mathbb{Z}/2N^2\mathbb{Z}$ such that $-D \equiv \beta^2 \pmod{4N^2}$

♣ Let

$$\mathcal{Q}_{D,(N),\beta} = \left\{[Na,\ b,\ Nc] \in \mathcal{Q}_D \mid b \equiv \beta \pmod{2N^2}\right\}.$$

on which the group $\Gamma_0^0(N)$ acts.

♣ There is a canonical bijection between

$$\mathcal{Q}_D/\Gamma(1) \quad \text{and} \quad \mathcal{Q}_{D,(N),\beta}/\Gamma_0^0(N),$$

for D not divisible as a discriminant by the squares of any prime dividing N.

♣ Let

$$L_1 = \left\{ X = \begin{pmatrix} b & 2Nc \\ 2Na & -b \end{pmatrix} \mid a,\ b,\ c\ \in \mathbb{Z} \right\}$$

be a lattice in the quadratic space.

♣ Assume that discriminant $-D \equiv \square \pmod{4N^2}$.

♣ For a $\Gamma_0^0(N)$-modular function f, we define

$$t_f^{(\beta)}(D) = \sum_{Q \in \mathcal{Q}_{D,(N),\beta}/\Gamma_0^0(N)} \frac{1}{|\overline{\Gamma}_0^0(N)_Q|} f(\tau_Q).$$

♣ Then we have

$$\mathrm{MT}_f^{L_1}(0,\ D) = 2 \sum_{\beta \in \mathbb{Z}/2N^2\mathbb{Z}} t_f^{(\beta)}(D).$$

♣ Using this, we can relate modular traces and modified Galois traces.

Modified Galois trace of class invariants

Let

- $-D$: an imaginary quadratic discriminant.
- $K = \mathbb{Q}(\tau_D)$: an imaginary quadratic field
- $f(\tau_D)$: a class invariant (i.e. $K(f(\tau_D)) = H_{\mathcal{O}_D}$).

♣ The modified Galois trace of $f(\tau_D)$ can be defined by

$$\mathrm{GT}_f(D) = \sum_{\mathcal{O} \supset \mathcal{O}_D} \frac{2}{w_{\mathcal{O}}} \sum_{\sigma \in \mathrm{Cl}(\mathcal{O})} (f(\tau_D))^{\sigma}.$$

where $w_{\mathcal{O}}$ is the number of unit elements in $\mathcal{O}$.

♣ Note that $w_{\mathcal{O}} = 2$ if $K \neq \mathbb{Q}(\sqrt{-1}),\ \mathbb{Q}(\sqrt{-3})$.

♣ Furthermore, if $\mathcal{O} \supset \mathcal{O}_D$ has discriminat $-d$, then $d|D$ and $-d \leq d_K$. So we can write

$$\mathrm{GT}_f(D) = \sum_{d|D,\ -d \leq d_K} \frac{2}{w_{\mathcal{O}_d}} \sum_{Q \in C(d)} (f(\tau_D))^{Q^{-1}}.$$

Examples

♣ Let

- $\gamma_2(\tau)$: the holomorphic cube root of j on $\mathbb{H}$, a modular function of level 3
- $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$: generators of $\mathrm{SL}_2(\mathbb{Z})$.

♣ It is well known that if $D > 4$ and $(3,\ D) = 1$, and if $B = 0$ for D even and $B = 1$ otherwise, then $\zeta_3^B \gamma_2(\tau_D)$ is a class invariant.

Using the actions of S and T given by

$$\gamma_2 \circ S = \gamma_2 \quad \text{and} \quad \gamma_2 \circ T = \zeta_3^{-1}\gamma_2.$$

and Shimura's reciprocity law, we have

$$\left(\zeta_3^B \gamma_2(\tau_D)\right)^{[3a,\ -b,\ 3c]} = \gamma_2(\tau_Q).$$

♣ Therefore, we obtain

$$\mathrm{GT}_{\zeta_3^B \gamma_2(D)} = \sum_{Q \in \mathcal{Q}_{D,(3),\beta}/\gamma_0^0(3)} \gamma_2(\tau_Q).$$

♣ Counting the number of possible β, we get

$$\mathrm{GT}_{\zeta_3^B \gamma_2(D)} = \frac{1}{4}\mathrm{MT}^{L_1}_{\gamma_2}(0,\ D).$$

♣ Using the above relation, Jeon, Kang and Kim proved that

$$q^{-1} + \sum_{\substack{D>0,\ (3,\ D)=1, \\ -D\equiv\square \pmod{36}}} \mathrm{GT}_{\zeta_3^B \gamma_2}(D) q^D$$

is a weakly holomorphic modular form of weight 3/2 on $\Gamma_0(36)$.

♣ This method can be applied to other ring class invariants or ray class invariants.

Outline

Let p be one or a prime and $\Gamma_0^+(p)$ be the group generated by the Hecke group $\Gamma_0(p)$ and the Fricke involution $W_p = \begin{pmatrix} 0 & -1/\sqrt{p} \\ \sqrt{p} & 0 \end{pmatrix}$.

Definition

Let k be a positive integer greater than 2 and let P_{k-2} denote the space of all polynomials of degree at most $k-2$. For $p \in \{1, 2, 3\}$ we define a subspace W_{k-2}^+ of P_{k-2} by

$$\begin{aligned} W_{k-2}^+ = \{g \in P_{k-2} \mid g + g|_{2-k}W_p = 0 \\ = g + g|_{2-k}U + g|_{2-k}U^2 + \cdots + g|_{2-k}U^{n_p-1}\} \end{aligned}$$

with $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $U = TW_p$, and $n_p = \begin{cases} 3, & \text{if } p = 1 \\ 2p, & \text{if } p = 2, 3 \end{cases}$. The elements of the space W_{k-2}^+ are called *period polynomials*.

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Period Polynomials

- When $p = 1, 2, 3,$ period polynomials have been investigated in relation to modular integrals, cusp forms via the Eichler-Shimura isomorphism and to various other areas of mathematics.
- The importance of period polynomials comes from their close connection with special values of modular L-functions.

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- The importance of period polynomials comes from their close connection with special values of modular L-functions.

Weakly holomorphic modular forms

For any even integer k and $\epsilon \in \{\pm 1\}$, let $M_k^{!,\epsilon}(p)$ be the space of weakly holomorphic modular forms (that is, meromorphic with poles only at the cusps) of weight k for $\Gamma_0(p)$ with $f|_k W_p = \epsilon f$. Each $f \in M_k^{!,\epsilon}(p)$ has a Fourier expansion of the form

$$f(z) = \sum_{n \geq n_0} a_f(n) q^n,$$

where $q = \exp(2\pi i z)$. We set $\mathrm{ord}_\infty f = n_0$ if $a_f(n_0) \neq 0$.

Basis for $M_k^{!,\epsilon}(p)$

- When the genus of $\Gamma_0^+(p)$ is zero, the space $M_k^{!,\epsilon}(p)$ has a canonical basis.
- When $p = 1$, such a canonical basis was constructed by Duke and Jenkins.
- Let m_k^ϵ denote the maximal order of a nonzero $f \in M_k^{!,\epsilon}(p)$ at ∞. Indeed, for every integer $m \geq -m_k^\epsilon$, there exists a unique weakly holomorphic modular form $\mathfrak{f}_{k,m}^\epsilon \in M_k^{!,\epsilon}(p)$ with Fourier expansion of the form

$$\mathfrak{f}_{k,m}^\epsilon(\tau) = q^{-m} + \sum_{n > m_k^\epsilon} a_k^\epsilon(m,n) q^n$$

  and together they form a basis for $M_k^{!,\epsilon}(p)$.
- If $k > 2$, the maximal order m_k^ϵ is given by $\dim S_k^\epsilon(p)$ where $S_k^\epsilon(p)$ denotes the space of holomorphic cusp forms f of weight k for $\Gamma_0(p)$ with $f|_k W_p = \epsilon f$.

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Eichler integrals and period polynomials

Definition

For $f = \sum_{n \gg -\infty} a_f(n) q^n \in M_k^{!,+}(p)$ we define the Eichler integral of f by

$$\mathcal{E}_f(z) := \sum_{\substack{n \gg -\infty \\ n \neq 0}} a_f(n) n^{1-k} q^n.$$

Definition

The period polynomial for f is defined by

$$r^+(f)(z) := c_k(\mathcal{E}_f - \mathcal{E}_f|_{2-k} W_p)(z) \quad \text{with} \quad c_k = -\frac{\Gamma(k-1)}{(2\pi i)^{k-1}},$$

which measures the obstruction to modularity of the Eichler integral of f.

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which measures the obstruction to modularity of the Eichler integral of f.

- Let $S_k^{!,+}(p)$ be the subspace of $M_k^{!,+}(p)$ consisting of weakly holomorphic modular forms for $\Gamma_0^+(p)$ with zero constant term in the Fourier expansion.
- Bol's identity states that for any $\gamma \in SL_2(\mathbb{R})$ and any function g, we have

$$\frac{d^{k-1}}{dz^{k-1}}(g|_{2-k}\gamma) = (\frac{d^{k-1}g}{dz^{k-1}})|_k\gamma, \tag{1}$$

from which it easily follows that $r^+(f)(z) \in P_{k-2}$ for each $f \in S_k^{!,+}(p)$.
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- Moreover r^+ defines a map from $S_k^{!,+}(p)$ to W_{k-2}^+.

Exact Sequence

For $p \in \{1,2,3\}$ and even $k > 2$, the following sequence is exact:

$$0 \longrightarrow D^{k-1}(M^{!,+}_{2-k}(p)) \longrightarrow S^{!,+}_k(p) \xrightarrow{r^+} \frac{W^+_{k-2}}{< (\sqrt{p}z)^{k-2} - 1 >} \longrightarrow 0$$

where D denotes the differential operator $\frac{1}{2\pi i}\frac{d}{dz}$. This gives an isomorphism

$$S^{!,+}_k(p)/D^{k-1}(M^{!,+}_{2-k}(p)) \cong W^+_{k-2}/< (\sqrt{p}z)^{k-2} - 1 >,$$

where the right hand side is isomorphic to $S^+_k(p) \bigoplus S^+_k(p)$ by the Eichler-Shimura theorem.

Thus we have

$$\dim S_k^{!,+}(p)/D^{k-1}(M_{2-k}^{!,+}(p)) = 2\dim S_k^{+}(p).$$

Question: Is there a nice basis for $S_k^{!,+}(p)/D^{k-1}(M_{2-k}^{!,+}(p))$ consisting of Hecke eigenforms?

Hecke operator

- For each positive integer n coprime to p, the Hecke operator T_n on the space of cusp forms for $\Gamma_0(p)$ defined in the usual way acts on $S_k^+(p)$.
- Common eigenforms of all Hecke operators T_n with n coprime to p are called *Hecke eigenforms*.
- The Hecke operators T_l with prime indices $l(\neq p)$ acting on $S_k^{!,+}(p)$ are defined in the same way. Indeed if $f \in S_k^{!,+}(p)$ has q-expansion $\sum a_f(n)q^n$, then

$$T_l f = l^{k/2-1} \sum_{\substack{ad=l \\ b\ (d)}} f|_k \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix} \right) = \sum (a_f(ln) + l^{k-1} a_f(\frac{n}{l}))q^n.$$

Weakly holomorphic Hecke eigenforms

Definition

Following Bringmann, Guerzhoy, Kent and Ono we call $f \in S_k^{!,+}(p)$ a *weakly holomorphic Hecke eigenform* with respect to $S_k^{!,+}(p)/D^{k-1}(M_{2-k}^{!,+}(p))$ if for every Hecke operator T_n with $(n,p)=1$ there is a complex number λ_n for which

$$T_n f - \lambda_n f \in D^{k-1}(M_{2-k}^{!,+}(p)).$$

Multiplicity Two Theorem

If $t = \dim S_k^+(p)$, then $\dim W_{k-2}^+ = 2t+1$ and $\dim S_k^{!,+}(p)/D^{k-1}(M_{2-k}^{!,+}(p)) = 2t$.

Theorem (2013, Bringmann, Guerzhoy, Kent and Ono)

When $p = 1$,

$$S_k^{!,+}(p)/D^{k-1}(M_{2-k}^{!,+}(p)) = \bigoplus \mathbb{T}_i$$

where each $\bigoplus \mathbb{T}_i$ *consists of a cuspidal Hecke eigenform and a weakly holomorphic Hecke eigenform with the same eigenvalues.*

Remark

- *The proof of the above theorem in [BGKO] uses the theory of harmonic Maass forms and Poincare series.*
- *We extended the above theorem to higher level cases to the primes for which* $\Gamma_0^+(p)$ *has genus zero (primes up to 71 excluding 37, 43, 53, 61 and 67).*
- *We give an explicit construction of basis of eigenforms which does not relying on the theory of harmonic Maass forms.*
- *We give an explicit description of the "polar" eigenform* h_n *in terms of a linear combination of cuspidal eigenforms* f_n *and the dual form* f_n^* *(Here the duality is with respect to a certain pairing of* f_n *introduced by Guerzhoy.*

Pairing

Let $f, g \in M_k^{!,+}(p)$ have Fourier expansions as follows:

$$f(z) = \sum_{n \gg -\infty} a_f(n) q^n \text{ and } g(z) = \sum_{n \gg -\infty} a_g(n) q^n.$$

Definition

Following Bringmann, Guerzhoy, Kent and Ono, we define a pairing $\{f, g\}$ by

$$\{f, g\} := \sum_{n \in \mathbb{Z}, n \neq 0} \frac{a_f(-n) a_g(n)}{n^{k-1}}. \tag{2}$$

It is antisymmetric, bilinear and Hecke equivariant. Specifically, for any prime $l (\neq p)$

$$\{T_l f, g\} = \{f, T_l g\}.$$

Quotient Space $\widehat{S_k^+(p)}$

- Following Guerzhoy, we set

$$\widehat{S_k^+(p)} := \frac{S_k^{!,+}(p)}{D^{k-1}(M_{2-k}^{!,+}(p)) \bigoplus S_k^+(p)}.$$

- If $t = \dim S_k^+(p)$, then $\dim \widehat{S_k^+(p)} = t$ and $\widehat{S_k^+(p)} =< \widehat{\mathfrak{f}_{k,1}}, \ldots, \widehat{\mathfrak{f}_{k,t}} >$.
- The pairing $\{\cdot,\cdot\}$ induces a non-degenerate pairing on $\widehat{S_k^+(p)} \times S_k^+(p)$.

Dual Basis

- Let $t = \dim S_k^+(p)$ and

$$\{f_n = \sum_{m>0} \lambda(n,m) q^m \mid n = 1, 2, \cdots, t\}$$

be a basis of $S_k^+(p)$ consisting of normalized Hecke eigenforms.

- Let

$$f_n^* = \sum_{m \geq -t} \mu(m,n) q^m$$

with $\mu(m,n) \in \mathbb{C}$ be a linear combination of $\mathfrak{f}_{k,1}, \ldots, \mathfrak{f}_{k,t}$ which is dual to f_n with respect to the pairing, i.e. $\{f_m^*, f_n\} = \delta_{mn}$ where δ_{mn} is the Kronecker delta function. Then such functions f_n^* are unique.

Dual Basis

Moreover $T_l f_n^*$ and $\lambda(n,l) f_n^*$ represent the same coset in $S_k^{!,+}(p)/(D^{k-1}(M_{2-k}^{!,+}(p)) \bigoplus S_k^+(p))$. Thus we can write

$$T_l f_n^* = \lambda(n,l) f_n^* + D^{k-1} g_{n,l} + \sum_{j=1}^{t} a_{jn}(l) f_j \tag{3}$$

for some $g_{n,l} \in M_{2-k}^{!,+}(p)$ and $a_{jn}(l) \in \mathbb{C}$.

Outline

Theorem

Let p be one or a prime number for which the genus of $\Gamma_0^+(p)$ is zero. Let $t = \dim S_k^+(p)$ and l be a prime different from p. Then the following assertions are true.
(i) Let $\Lambda_1 = ((\lambda_1)_{ij})$ and $\Lambda_2 = ((\lambda_2)_{ij})$ be $t \times t$ matrices whose ij-entries are given by $(\lambda_1)_{ij} = \lambda(i,j)$ and $(\lambda_2)_{ij} = \lambda(i,j)/j^{k-1}$. Then the following relations are satisfied:

$$\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_t \end{pmatrix} = \Lambda_1 \begin{pmatrix} \mathfrak{f}_{k,-1} \\ \mathfrak{f}_{k,-2} \\ \vdots \\ \mathfrak{f}_{k,-t} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} f_1^* \\ f_2^* \\ \vdots \\ f_t^* \end{pmatrix} = ((\Lambda_2)^{-1})^T \begin{pmatrix} \mathfrak{f}_{k,1} \\ \mathfrak{f}_{k,2} \\ \vdots \\ \mathfrak{f}_{k,t} \end{pmatrix}.$$

Theorem (continued)

(ii) Let $m, n \in \{1, \cdots, t\}$. The quantity $a_{mn}(l)$ in (3) is computed in terms of the pairing as follows:

$$a_{mn}(l) = -\{T_l f_n^*, f_m^*\}.$$

Moreover $a_{mn}(l) = -a_{nm}(l)$ and $a_{nn}(l) = 0$.
(iii) Let $i, n \in \{1, \cdots, t\}$ with $i \neq n$. Let r be a prime ($\neq p$) such that $\lambda(i, r) \neq \lambda(n, r)$ and put

$$x_i(n) := \frac{a_{ni}(r)}{\lambda(i, r) - \lambda(n, r)}.$$

Then $x_i(n)$ is independent of the choice of r.

Theorem (continued)

(iv) For each n with $1 \leq n \leq t$, let

$$h_n := \sum_{\substack{i=1 \\ i \neq n}}^{t} x_i(n) f_i + f_n^*.$$

Then h_n is a Hecke eigenform with respect to $S_k^{!,+}(p)/D^{k-1}(M_{2-k}^{!,+}(p))$ having the same eigenvalues as those of f_n. More explicitly one has

$$T_l(h_n) = \lambda(n,l) h_n + D^{k-1}(g_{n,l})$$

where $g_{n,l}$ is the modular form defined in (3) and computed as

$$g_{n,l} = - \sum_{\substack{1 \leq s \leq t \\ sl > t}} \frac{\mu(-s,n)}{s^{k-1}} f_{2-k,sl}$$

Theorem (continued)

where $\mu(\cdot, n)$ *is the Fourier coefficient of* f_n^*.
(v) The set

$$\{[f_1], \cdots, [f_t], [h_1], \cdots, [h_t]\}$$

forms a basis for $S_k^{!,+}(p)/D^{k-1}(M_{2-k}^{!,+}(p))$ *where* $[f]$ *stands for the class of* f.

Algebraicity of dual forms

Theorem

Let $p \in \{1, 2, 3, 5, 7, 13\}$, $2 < k \in 2\mathbb{Z}$, $t = \dim S_k^+(p)$, *and* $n \in \{1, \cdots, t\}$. *Then the coefficients of* f_n^* *are in* K_{f_n}.

Example of algebraicity of dual forms

Let $p = 5$ and $k = 12$. In this case $t = \dim S_{12}^{+}(5) = 3$ and the space $S_{12}^{+}(5)$ is spanned by

$$f_{12,-3}(z) = \Delta_5^{+}(z)^3 = (\eta(z)\eta(5z))^{12} = q^3 - 12q^4 + \cdots \ ,$$
$$f_{12,-2}(z) = \Delta_5^{+}(z)^3(j_5^{+}(z) + 12) = q^2 + 44q^4 - 288q^5 + 306q^6 + \cdots \ ,$$
$$f_{12,-1}(z) = \Delta_5^{+}(z)^3(j_5^{+}(z)^2 + 12j_5^{+}(z) - 178) = q + 2608q^4 + \cdots \ .$$

The Hecke eigenforms are given by

$$f_1 = f_{12,-1} - 24f_{12,-2} + 252f_{12,-3},$$
$$f_2 = f_{12,-1} + (-10 + 6\sqrt{151})f_{12,-2} + (-110 + 32\sqrt{151})f_{12,-3},$$
$$f_3 = f_{12,-1} + (-10 - 6\sqrt{151})f_{12,-2} + (-110 - 32\sqrt{151})f_{12,-3},$$

so that $K_{f_1} = \mathbb{Q}$ and $K_{f_2} = K_{f_3} = \mathbb{Q}(\sqrt{151})$.

Example of algebraicity of dual forms

The dual forms are given by

$$
\begin{aligned}
f_1^* &= \frac{17}{131} f_{12,1} - \frac{16384}{655} f_{12,2} + \frac{531441}{1310} f_{12,3}, \\
f_2^* &= \frac{3(2869 + 43\sqrt{151})}{19781} f_{12,1} + \frac{512(2416 + 181\sqrt{151})}{98905} f_{12,2} \\
&\quad + \frac{177147(-453 + 7\sqrt{151})}{395620} f_{12,3}, \\
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so that $K_{f_i^*} = K_{f_i}$ for each $i \in \{1, 2, 3\}$.

Outline

1. Zagier lifts
   - Modular function field
   - Class fields over an imaginary quadratic field
   - Shimura reciprocity law
   - Theta lift
   - Modularity of Galois traces

2. Weakly holomorphic Hecke eigenforms
   - Construction of Weakly holomorphic Hecke eigenforms in higher level cases
   - Explicit construction of mock modular forms

Weak Maass form

Definition

A *weak Maass form of weight k on a congruence subgroup* Γ is any smooth function $f : \mathbb{H} \to \mathbb{C}$ satisfying:

1. For all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and $z \in \mathbb{H}$, we have

$$f(\gamma z) = (\det \gamma)^{-k/2}(cz+d)^k f(z)$$

2. We have that $\Delta_k f := [-y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)]f = \lambda f$ for some $\lambda \in \mathbb{C}$.
3. The function $f(z)$ has at most linear exponential growth at all cusps.

A weak Maass form f is called harmonic if $\Delta_k f = 0$.

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Differential Operator

Bruinier-Funke (2004)
If h is a harmonic weak Maass form of weight k, then it follows from $\Delta_k(h) = 0$ and

$$\Delta_k = -\xi_{2-k} \circ \xi_k, \quad \text{where} \quad \xi_w := 2iy^w \overline{\frac{\partial}{\partial \overline{z}}}$$

that $\xi_k(h) \in M^!_{2-k}$.

$H_k(\Gamma_0(N)) :=$ the space of harmonic weak Maass forms h of weight k on $\Gamma_0(N)$ satisfying $\xi_k(h) \in S_{2-k}(N)$.

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Good Maass form

Definition

$\mathfrak{F}(z) \in H_{2-k}(\Gamma_0(p))$ is called "good" for the Hecke eigenform $f^c(z) := \overline{f(-\bar{z})} \in S_k(p)$ if it satisfies the following:

1. The principal part of $\mathfrak{F}$ at the cusp ∞ belongs to $K_f[q^{-1}]$. Here K_f denotes the number field obtained by adjoining to $\mathbb{Q}$ the Fourier cofficients of f.
2. The principal part of $\mathfrak{F}$ at the cusp 0 is constant.
3. We have $\xi_{2-k}\mathfrak{F} = \frac{f^c}{(f^c, f^c)}$.

Good Maass form

- The existence of $\mathfrak{F}$, which is good for a Hecke eigenform f^c is garanteed by Bruinier, Ono and Rhoades' work.
- Let $\mathfrak{F}_0$ be good for a Hecke eigenform f^c and denote $t = \dim S_k^+(p)$ and $t' = \dim S_k^-(p)$. Let $M_{2-k}^\sharp(p)$ be the space of weakly holomorphic modular forms of weight $2-k$ for $\Gamma_0(p)$ with poles allowed only at the cusp ∞. For $p \in \{1, 2, 3, 5, 7, 13\}$, it follows from Ahn and Choi's work that

$$\max\{\mathrm{ord}_\infty f \mid f \neq 0 \in M_{2-k}^\sharp(p)\} = -1 - t - t',$$

  and for each integer m with $-m \leq -1 - t - t'$, there exists $\mathfrak{f}_{2-k,m}^\sharp = q^{-m} + O(q^{-t-t'}) \in M_{2-k}^\sharp(p)$ with integral Fourier coefficients. By subtracting a suitable linear combination of $\mathfrak{f}_{2-k,m}^\sharp$'s from $\mathfrak{F}_0$ we can take a unique $\mathfrak{F}$ which is good for f^c and $\mathfrak{F}^+ = O(q^{-t-t'})$.

Good Maass form

- Guerzhoy, Kent and Ono provided a direct method for relating the coefficients of f^c and $\mathfrak{F}$ by means of p-adic coupling and an algebraic regularized mock modular form $\mathfrak{F}_\alpha^+$. More precisely, if we let α be the coefficient of q^1 in $\mathfrak{F}^+$, then $F_\alpha := D^{k-1}\mathfrak{F}_\alpha^+$ is given by

$$F_\alpha = D^{k-1}\mathfrak{F}^+ - \alpha f = \sum_{n \gg -\infty} c_\alpha(n) q^n.$$

- Moreover F_α has coefficients in K_f and

$$(*) \qquad \lim_{w \to +\infty} \frac{\sum_{n \gg -\infty} c_\alpha(l^w n) q^n}{c_\alpha(l^w)} = f(z) - \beta^{-1} l^{k-1} f(lz).$$

Here l is a prime number and β, β' are the roots of the equation $X^2 - a_f(l)X + \chi(l)l^{k-1} = (X-\beta)(X-\beta')$ ordered so that $\mathrm{ord}_l(\beta) \le \mathrm{ord}_l(\beta')$, χ is a trivial character modulo p, and we assume that $\beta \neq 0$ in the case $l = p$.

Good Maass form

- The structure of half-integral weight weakly holomorphic Hecke eigenforms was developed and half-integral weight p-adic coupling was investigated by Bringmann, Guerzhoy and Kane.

Regularized inner product

- For $T > 0$, we denote by $\mathcal{F}_T$ the truncated fundamental domain for $SL_2(\mathbb{Z})$

$$\mathcal{F}_T = \{z \in \mathbb{H} \mid |x| \leq 1/2, |z| \geq 1, \text{ and } y \leq T\}.$$

- Moreover, we define the truncated fundamental domain for $\Gamma_0(p)$ by

$$\mathcal{F}_T(\Gamma_0(p)) = \bigcup_{\gamma \in \Gamma_0(p) \backslash SL_2(\mathbb{Z})} \gamma \mathcal{F}_T.$$

- For $f, g \in M_k^!(p)$, we define the regularized inner product $(f,g)^{\text{reg}}$ as the constant term in the Laurent expansion at $s = 0$ of the function

$$\frac{1}{[SL_2(\mathbb{Z}) : \Gamma_0(p)]} \lim_{T \to \infty} \int_{\mathcal{F}_T(\Gamma_0(p))} f(z)\overline{g(z)} y^{k-s} \frac{dxdy}{y^2}.$$

Regularized inner product

- By Borcherds and independently Bruinier, Ono, and Rhoades, $(f, g)^{\mathrm{reg}}$ exists if f or g is a holomorphic modular form. If both f and g are holomorphic modular forms such that fg is a cusp form, then $(f, g)^{\mathrm{reg}}$ reduces to the Petersson inner product (f, g).

Good Maass form of level 1

Theorem

Let $2 < k \in 2\mathbb{Z}$, *and* $t = \dim S_k^+(1)$. *Then for each* $n \in \{1, \cdots, t\}$,

$$-h_n + \frac{(f_n^*, f_n)^{\text{reg}}}{(f_n, f_n)} f_n$$

is equal to $D^{k-1}\mathfrak{F}$ *for a unique* $\mathfrak{F} \in H_{2-k}(\Gamma_0(1))$ *which is good for* f_n^c *and* $\mathfrak{F}^+ = O(q^{-t})$.

Example of good Maass form of level 1

Let $p = 1$ and $k = 12$. In this case we have $t = \dim S_{12}^{+}(1) = 1$. Using $\Delta(z) = \eta(z)^{24} \in S_{12}^{+}(1)$ and the Hauptmodul $j_1(z) = E_4(z)^3/\Delta(z) - 744$ for $\Gamma_0^{+}(1)$ one can express $f_{12,m}$ $(-1 \le m \le 1)$ as follows:

$$f_{12,-1}(z) = \Delta(z) = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 + \cdots$$
$$f_{12,0}(z) = \Delta(j_1 + 24) = 1 + 196560q^2 + 16773120q^3 + \cdots$$
$$f_{12,1}(z) = \Delta({j_1}^2 + 24j_1 - 393444) = \frac{1}{q} + 47709536q^2 + \cdots \quad .$$

Example of good Maass form of level 1

$$f_1 = f_{12,-1} \quad \text{and} \quad h_1 = f_1^* = f_{12,1}$$

$$F_\alpha = -h_1 = -f_{12,1} = -q^{-1} - \sum_{n=2}^{\infty} a_{12}(1,n)q^n,$$

If we take $l = 3$ and $w = 1$ in (*), by using Sturm bound one verifies that

$$\begin{aligned}
&\frac{-\sum_{n=1}^{\infty} a_{12}(1,3n)q^n}{-a_{12}(1,3)} \\
&= q + \frac{27947672851540608}{39862705122}q^2 + \frac{340389905850815087232}{39862705122}q^3 + \cdots \\
&\equiv \Delta \pmod{3^{10}}.
\end{aligned}$$

Good Maass form of higher level

Theorem (S. Choi and K.)

$p \in \{2,3,5,7,13\}$, $2 < k \in 2\mathbb{Z}$, $t = \dim S_k^+(p)$, $t' = \dim S_k^-(p)$

1. *Let A be a $t \times t$ matrix whose ij-entry is given by $CT(f_i \cdot f_{2-k,t'+j}^-)$ where $CT(f)$ denotes the constant term of the Fourier expansion of f. Then the matrix A is invertible.*
2. *Let β_{ij} be the ij-entry of the matrix A^{-1}. Take a unique weakly holomorphic modular form $w_n \in M_{2-k}^{!+}(p)$ such that $w_n - \sum_{j=1}^t \beta_{jn} f_{2-k,t'+j}^- \in O(q^{-t})$. Then*
$$-h_n + \frac{(f_n^*, f_n)^{\mathrm{reg}}}{(f_n, f_n)} f_n + D^{k-1}\Big(w_n + \sum_{j=1}^{t} \beta_{jn} f_{2-k,t'+j}^-\Big)$$
is equal to $D^{k-1}\mathfrak{F}$ for a unique $\mathfrak{F} \in H_{2-k}(\Gamma_0(p))$ which is good for f_n^c and $\mathfrak{F}^+ = O(q^{-t-t'})$.

Example of good Maass form of level 5

$p=5;\ k=10 \Longrightarrow t=\dim S_{10}^{+}(5)=1,\ t'=\dim S_{10}^{-}(5)=2.$
$E_6^{+} := \frac{1}{1+5^3}(E_6 + E_6|_6 W_5),\ E_4^{-} := \frac{1}{1-5^2}(E_4 - E_4|_4 W_5),$

$$f_{10,-1} = \Delta_5^{+} E_6^{+} = q - 8q^2 - 114q^3 - 448q^4 - 625q^5 + 912q^6 + \cdots\ ,$$

$$f_{10,1} = \Delta_5^{+} E_6^{+} ({j_5^{+}}^2 + 8j_5^{+} - 90) = \frac{1}{q} - 192q^2 - 14511q^3 + \cdots\ ,$$

$$f_{-8,2} = (\Delta_5^{+})^{-2} = \frac{1}{q^2} + \frac{8}{q} + 44 + 192q + 726q^2 + 2472q^3 + \cdots\ ,$$

$$f_{-8,3} = (\Delta_5^{+})^{-2}(j_5^{+} - 8) = \frac{1}{q^3} + \frac{114}{q} + 1672 + 14511q + \cdots\ ,$$

$$f_{-8,3}^{-} = (\Delta_5^{+})^{-3} E_4^{-} = \frac{1}{q^3} + \frac{2}{q^2} - \frac{120}{q} - 1740 - 14855q + \cdots\ ,$$

$$f_1 = f_{10,-1},\ \ h_1 = f_1^{*} = f_{10,1} \quad \text{and} \quad w_1 = \beta_{11} f_{-8,3} + 2\beta_{11} f_{-8,2},$$

where $\beta_{11} = \frac{1}{CT(f_1 \cdot f_{-8,3}^{-})} = -\frac{1}{250}.$

Example of good Maass form of level 5

$$F_\alpha = -h_1 + D^9(w_1 + \beta_{11} f^-_{-8,3}) + \frac{4}{25} f_1 = \sum_{n=-3}^{\infty} c_\alpha(n) q^n$$

Take $l = 3$ and $w = 1$ in (*). Using Sturm bound one verifies that

$$\begin{aligned}
&\frac{\sum_{n=-1}^{\infty} c_\alpha(3n) q^n}{c_\alpha(3)} \\
&= -\frac{6561}{6308q} - \frac{18528264}{1577} q^2 + \frac{808269273}{1577} q^3 + \frac{68622811200}{1577} q^4 + \cdots \\
&\equiv f_1 \pmod{3^8}.
\end{aligned}$$

Thank you for your attention.