Erratum : A Geometrical Theory of Jacobi Forms of Higher Degree

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Erratum

In the article *A Geometrical Theory of Jacobi Forms of Higher Degree* by Jae-Hyun Yang [Kyungpook Math. J., **40(2)**(2000), 209-237], the author presents the Laplace-Beltrami operator $\Delta_{g,h}$ of the Siegel-Jacobi space $(H_{g,h},\ ds^2_{g,h})$ given by the formula (10.4) without a proof at the page 227. But the operator $\Delta_{g,h}$ is **not a correct one**.

At the page 227, the formula (10.4) should be replaced by the following **correct formula (10.4)**:

$$
(10.4) \quad \Delta_{g,h} \;=\; 4\,\sigma\!\left(Y\,{}^t\!\Big(Y\frac{\partial}{\partial \overline{Z}}\Big)\frac{\partial}{\partial Z} \right) + 4\,\sigma\!\left(Y\frac{\partial}{\partial W}\,{}^t\!\Big(\frac{\partial}{\partial \overline{W}}\Big) \right)
$$
$$
+\,4\,\sigma\!\left(V Y^{-1}\,{}^tV\,{}^t\!\Big(Y\frac{\partial}{\partial \overline{W}}\Big)\frac{\partial}{\partial W} \right)
$$
$$
+\,4\,\sigma\!\left(V\,{}^t\!\Big(Y\frac{\partial}{\partial \overline{Z}}\Big)\frac{\partial}{\partial W} \right) + 4\,\sigma\!\left({}^tV\,{}^t\!\Big(Y\frac{\partial}{\partial \overline{W}}\Big)\frac{\partial}{\partial Z} \right).
$$

A Geometrical Theory of Jacobi Forms of Higher Degree

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In this paper, we give a survey of a geometrical theory of Jacobi forms of higher degree. And we present some geometric results and discuss some geometric problems to be investigated in the future.

1. Introduction

A Jacobi form is an automorphic form on the Jacobi group, which is the semi-direct product of the symplectic group $Sp(g, \mathbb{R})$ and the Heisenberg group $H_{\mathbb{R}}^{(g,h)}$ (see section 2). Jacobi forms are very useful because they are closely related to modular forms of half integral weight and the theory of the moduli space of abelian varieties. The simplest case is when the symplectic group is $SL(2, \mathbb{R})$ and the Heisenberg group is three dimensional, that is, $g = h = 1$. This case had been treated more or less systematically in [21] and many papers of Zagier's school. But it seems to us that there is no systematic investigation of Jacobi forms of higher degree when $g > 1$ and $h > 1$. Some results could be found in [17], [79]-[89] and [94].

The purpose of this paper is to give a survey of a geometrical theory of Jacobi forms of higher degree. And we present some geometric results and discuss some geometric problems which should be investigated in the future. In Section 2, we review the notion of Jacobi forms and establish the notations. In Section 3, we present a brief historical remark and some motivation on Jacobi forms. In Section 4, we review the toroidal compactifications of the Siegel modular variety and the universal abelian variety. In Section 5, we introduce the automorphic vector bundle $E_{\rho, \mathcal{M}}$ associated with the canonical automorphic factor $J_{\mathcal{M}, \rho}$ for the Jacobi group $G_{g,h}^J$ and then discuss the properties of $E_{\rho, \mathcal{M}}$ related to Jacobi forms. In Section 6, we give some open problems related to Wang's result(cf. [63]). In Section 7, we describe the boundary of the Satake compactification in terms of the languages of Jacobi forms. These results are essentially due to Igusa [35]. In Section 8, we

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provide you with some characterizations of *singular* Jacobi forms due to Yang [85]. We roughly explain that the study of *singular* Jacobi forms is closely related to the invariant theory of the action of the group $GL(g, \mathbb{R}) \ltimes H_{\mathbb{R}}^{(g,h)}$(cf. (9.1)) and to the geometry of the universal abelian variety. In Section 9, we introduce some results of the Siegel-Jacobi operator. We describe implicitly that the Siegel-Jacobi operator plays an important role in the study of the universal abelian variety. In Section 10, we present $G_{g,h}^{J}$-invariant Kähler metrics and $G_{g,h}^{J}$-invariant differential operators on the Siegel-Jacobi space $H_g \times \mathbb{C}^{(h,g)}$. We introduce the notion of Maass-Jacobi forms. In the final section, we give a brief remark on some recent geometric results. In appendix A, we talk about subvarieties of the Siegel modular variety and present several problems. In appendix B, we describe why the study of *singular modular forms* is closely related to that of the geometry of the Siegel modular variety. Finally I would like to give my hearty thanks to Professor Tadao Oda and Dr. Hiroyuki Ito for inviting me to Sendai and giving me a chance to give a lecture at the conference on Hodge Theory and Algebraic Geometry.

Notations: We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of integers, the field of real numbers, and the field of complex numbers respectively. H_g denotes the Siegel upper half plane of degree g. $\Gamma_g := Sp(g, \mathbb{Z})$ denotes the Siegel modular group of degree g. The symbol ":=" means that the expression on the right is the definition of that on the left. We denote by $\mathbb{Z}^+$ the set of all positive integers. $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F. For a square matrix $A \in F^{(k,k)}$ of degree k, $\sigma(A)$ denotes the trace of A. For $A \in F^{(k,l)}$ and $B \in F^{(k,k)}$, we set $B[A] = {}^tABA$. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transpose matrix of M. E_n denotes the identity matrix of degree n. For a commutative ring K, we denote by $S_\ell(K)$ the vector space of symmetric matrices of degree ℓ with entries in K. For a positve integer g and an integer k, we denote by $[\Gamma_g, k]$ the vector space of all Siegel modular forms on H_g of weight k.

2. Jacobi Forms

In this section, we establish the notations and define the concept of Jacobi forms.

Let
$$Sp(g, \mathbb{R}) = \{M \in \mathbb{R}^{(2g,2g)} \mid {}^tMJ_gM = J_g \}$$
be the symplectic group of degree g, where
$$J_g := \left(\begin{pmatrix} 0 & E_g \\ -E_g & 0 \end{pmatrix} \right).$$
It is easy to see that $Sp(g, \mathbb{R})$ acts on H_g transitively by
$$M < Z >:= (AZ + B)(CZ + D)^{-1},$$
where $M = \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) \in Sp(g, \mathbb{R})$ and $Z \in H_g$.

For two positive integers g and h, we recall that the Jacobi group $G_{g,h}^J :=$ $Sp(g, \mathbb{R}) \ltimes H_{\mathbb{R}}^{(g,h)}$ is the semidirect product of the symplectic group $Sp(g, \mathbb{R})$ and the Heisenberg group $H_{\mathbb{R}}^{(g,h)}$ endowed with the following multiplication law

$$(M, (\lambda, \mu, \kappa)) \cdot (M', (\lambda', \mu', \kappa')) := (MM', (\tilde{\lambda} + \lambda', \tilde{\mu} + \mu', \kappa + \kappa' + \tilde{\lambda}\,{}^t\mu' - \tilde{\mu}\,{}^t\lambda'))$$

with $M, M' \in Sp(g, \mathbb{R}), (\lambda, \mu, \kappa), (\lambda', \mu', \kappa') \in H_{\mathbb{R}}^{(g,h)}$ and $(\tilde{\lambda}, \tilde{\mu}) := (\lambda, \mu)M'$. It is easy to see that $G_{g,h}^J$ acts on $H_{g,h} := H_g \times \mathbb{C}^{(h,g)}$ transitively by

$$(2.1) \qquad (M, (\lambda, \mu, \kappa)) \cdot (Z, W) := (M < Z >, (W + \lambda Z + \mu)(CZ + D)^{-1}),$$

where $M = \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) \in Sp(g, \mathbb{R}), \ (\lambda, \mu, \kappa) \in H_{\mathbb{R}}^{(g,h)}$ and $(Z, W) \in H_{g,h}$.

Let ρ be a rational representation of $GL(g, \mathbb{C})$ on a finite dimensional complex vector space V_ρ. Let $\mathcal{M} \in \mathbb{R}^{(h,h)}$ be a symmetric half-integral semi-positive definite matrix of degree h. Let $C^\infty(H_{g,h}, V_\rho)$ be the algebra of all C^∞ functions on $H_{g,h}$ with values in V_ρ. For $f \in C^\infty(H_{g,h}, V_\rho)$, we define

$$
\begin{aligned}
(2.2) \qquad & (f|_{\rho, \mathcal{M}}[(M, (\lambda, \mu, \kappa))])(Z, W) \\
:= \quad & e^{-2\pi i \sigma(\mathcal{M}[W + \lambda Z + \mu](CZ + D)^{-1}C)} \times e^{2\pi i \sigma(\mathcal{M}(\lambda Z^t \lambda + 2\lambda^t W + (\kappa + \mu^t \lambda)))} \\
& \times \rho(CZ + D)^{-1} f(M < Z >, (W + \lambda Z + \mu)(CZ + D)^{-1}),
\end{aligned}
$$

where $M = \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) \in Sp(g, \mathbb{R}), \ (\lambda, \mu, \kappa) \in H_{\mathbb{R}}^{(g,h)}$ and $(Z, W) \in H_{g,h}$.

Definition 2.1. Let ρ and $\mathcal{M}$ be as above. Let

$$H_{\mathbb{Z}}^{(g,h)} := \{ (\lambda, \mu, \kappa) \in H_{\mathbb{R}}^{(g,h)} \,|\, \lambda, \mu \in \mathbb{Z}^{(h,g)}, \ \kappa \in \mathbb{Z}^{(h,h)} \}.$$

Let Γ be a discrete subgroup of Γ_g of finite index. A *Jacobi form* of index $\mathcal{M}$ with respect to ρ on Γ is a holomorphic function $f \in C^\infty(H_{g,h}, V_\rho)$ satisfying the following conditions (A) and (B):

(A) $f|_{\rho, \mathcal{M}}[\tilde{\gamma}] = f$ for all $\tilde{\gamma} \in \Gamma^J := \Gamma \ltimes H_{\mathbb{Z}}^{(g,h)}$.

(B) f has a Fourier expansion of the following form :

$$f(Z, W) = \sum_{\substack{T \geq 0 \\ \text{half-integral}}} \sum_{R \in \mathbb{Z}^{(g,h)}} c(T, R) \cdot e^{\frac{2\pi i}{\lambda_\Gamma} \sigma(TZ)} \cdot e^{2\pi i \sigma(RW)}$$

with some nonzero integer $\lambda_\Gamma \in \mathbb{Z}$ and $c(T, R) \neq 0$ only if $\left(\begin{pmatrix} \frac{1}{\lambda_\Gamma} T & \frac{1}{2} R \\ \frac{1}{2}{}^t R & \mathcal{M} \end{pmatrix} \right) \geq 0$.

If $g \geq 2$, the condition (B) is superfluous by Köcher principle (cf. [94] Lemma 1.6). We denote by $J_{\rho, \mathcal{M}}(\Gamma)$ the vector space of all Jacobi forms of index $\mathcal{M}$ with respect to ρ on Γ. Ziegler(cf. [94] Theorem 1.8 or [21] Theorem 1.1) proves that the

vector space $J_{\rho,\mathcal{M}}(\Gamma)$ is finite dimensional. For more results on Jacobi forms with $g > 1$ and $h > 1$, we refer to [17], [79]-[89] and [94].

Definition 2.2. A Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma)$ is said to be a *cusp* (or *cuspidal*) form if $\begin{pmatrix} \frac{1}{\lambda_\Gamma}T & \frac{1}{2}R \\ \frac{1}{2}{}^tR & \mathcal{M} \end{pmatrix} > 0$ for any T, R with $c(T,R) \neq 0$. A Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma)$ is said to be *singular* if it admits a Fourier expansion such that a Fourier coefficient $c(T,R)$ vanishes unless $\det \begin{pmatrix} \frac{1}{\lambda_\Gamma}T & \frac{1}{2}R \\ \frac{1}{2}{}^tR & \mathcal{M} \end{pmatrix} = 0$.

Example 2.3. Let $S \in \mathbb{Z}^{(2k,2k)}$ be a symmetric, positive definite, unimodular even integral matrix and $c \in \mathbb{Z}^{(2k,h)}$. We define the theta series

$$(2.2) \qquad \vartheta_{S,c}^{(g)}(Z,W) := \sum_{\lambda \in \mathbb{Z}^{(2k,g)}} e^{\pi i\{\sigma(S\lambda Z \,{}^t\lambda) + 2\sigma({}^tcS\lambda \,{}^tW)\}}, \quad Z \in H_g, \; W \in \mathbb{C}^{(h,g)}.$$

We put $\mathcal{M} := \frac{1}{2}{}^tcSc$. We assume that $2k < g + rank\,(\mathcal{M})$. Then it is easy to see that $\vartheta_{S,c}^{(g)}$ is a singular Jacobi form in $J_{k,\mathcal{M}}(\Gamma_g)$(cf. [94] p.212).

3. Historical Remarks

In this section, we will make brief historical remarks on Jacobi forms.

In 1985, the names Jacobi group and Jacobi forms got kind of standard by the classic book [21] by EICHLER and ZAGIER to remind of Jacobi's "Fundamenta nova theoriae functionum ellipticorum", which appeared in 1829 ([36]). Before [21] these objects appeared more or less explicitly and under different names in the work of many authors.

In 1969 Pyatetski-Shapiro [52] discussed the Fourier-Jacobi expansion of Siegel modular forms and the field of modular abelian functions. He gave the dimension of this field in the higher degree.

About the same time Satake [55]-[56] introduced the notion of "groups of Harish-Chandra type" which are non reductive but still behave well enough so that he could determine their canonical automorphic factors and kernel functions.

Shimura [57]-[58] gave a new foundation of the theory of complex multiplication of abelian functions using Jacobi theta functions.

Kuznetsov [41] constructed functions which are almost Jacobi forms from ordinary elliptic modular functions.

Starting 1981, Berndt [4]-[6] published some papers which studied the field of arithmetic Jacobi functions, ending up with a proof of Shimura reciprocity law for the field of these functions with arbitrary level. Furthermore he investigated the discrete series for the Jacobi group $G_{g,h}^J$ and developed the spectral theory for $L^2(\Gamma^J\backslash G_{g,h}^J)$ in the case $g = h = 1$([9],[11]). Recently he [10] studied the L-

functions and the Whittaker models for the Jacobi forms.

The connection of Jacobi forms to modular forms was given by Maass, Andrianov, Kohnen, Shimura, Eichler and Zagier. This connection is pictured as follows. For k even, we have the following isomorphisms

$$[\Gamma_2, k]^M \cong J_{k,1}(\Gamma_1) \cong M^+_{k-\frac{1}{2}}(\Gamma_0^{(1)}(4)) \cong [\Gamma_1, 2k-2].$$

Here $[\Gamma_2, k]^M$ denotes the Maass's Spezialschar, $M^+_{k-\frac{1}{2}}(\Gamma_0^{(1)}(4))$ denotes the Kohnen space and $[\Gamma_1, 2k-2]$ denotes the vector space consisting of elliptic modular forms of weight $2k-2$. For a precise detail, we refer to [42]-[44], [1], [21], [37] and [81].

In 1982 Tai [60] gave asymptotic dimension formulae for certain spaces of Jacobi forms for arbitrary g and $h = 1$ and used these ones to show that the moduli A_g of principally polarized abelian varieties of dimension g is *of general type* for $g \geq 9$.

Feingold and Frenkel [23] essentially discussed Jacobi forms in the context of Kac-Moody Lie algebras generalizing the Maass correspondence to higher level. Gritsenko [30] studied Fourier-Jacobi expansions and a non-commutative Hecke ring in connection with the Jacobi group.

After 1985 the theory of Jacobi forms for $g = h = 1$ had been studied more or less systematically by the Zagier school. A large part of the theory of Jacobi forms of higher degree was investigated by Dulinski [17], Kramer [40], Yamazaki [69], Yang [79]-[89] and Ziegler [94].

There were several attempts to establish L-functions in the context of the Jacobi group by Murase [47]-[48] and Sugano [50] using the so-called "Whittaker-Shintani functions".

Recently Kramer [40] developed an arithmetic theory of Jacobi forms of higher degree. Runge [54] discussed some part of the geometry of Jacobi forms for arbitrary g and $h = 1$. Quite recently T. Arakawa and B. Heim [2] studied the iterated Petersson scalar product of a diagonal-restricted real analytic Jacobi Eisenstein series of degree (3,1) against elliptic Jacobi forms generalizing Garrett's result in the case of Siegel Eisenstein series of degree 3.

For a good survey on some motivation and background for the study of Jacobi forms, we refer to [10].

4. Review on Toroidal Compactifications of the Siegel Space and the Universal Abelian Variety

In this section, we will make a brief review on toroidal compactification of the Siegel space and the universal abelian variety. We refer to [3], [22] and [51] for more detail.

I. A toroidal compactification of the Siegel modular variety

First we realize H_g as a bounded symmetric domain $D_g := \{\, W \in \mathbb{C}^{(g,g)} \mid W = {}^t W, \ E_g - Z\bar{Z} > 0 \,\}$ (called the generalized unit disc of degree g) in $S_g(\mathbb{C})$ via the transformation $\Phi : H_g \longrightarrow D_g$ given by

$$\Phi(Z) := (Z - iE_g)(Z + iE_g)^{-1}, \quad Z \in H_g.$$

Indeed, it is a Harish-Chandra realization of a homogeneous space. The inverse Φ^{-1} of Φ given by

$$\Phi^{-1}(Z) := i\,(E_g + W)(E_g - W)^{-1}, \quad W \in D_g$$

is called the *generalized Cayley transformation.*

Let $\bar{D}_g$ be the topological closure of D_g in $S_g(\mathbb{C})$. Then $\bar{D}_g$ is the disjoint union of all boundary components of D_g. Let

$$F_r := \left\{ \begin{pmatrix} Z_1 & 0 \\ 0 & E_{g-r} \end{pmatrix} \in \bar{D}_g \mid Z_1 \in D_r \right\}, \quad 0 \le r \le g$$

be the standard rational boundary components of D_g. Then any boundary component F of D_g is of the form $F = g \cdot F_r$ for some $g \in Sp(g,\mathbb{R})$ and some r with $0 \le r \le g$. In addition, if F is a rational boundary component of D_g, then it is of the form $F = \gamma \cdot F_r$ for some $\gamma \in Sp(g,\mathbb{Z})$ and some r with $0 \le r \le g$. We note that $F_0 = \{\, E_g \,\}$ and $F_g = D_g$. We set

$$(4.1) \qquad\qquad D_g^* := \cup_{0 \le r \le g}\, Sp(g,\mathbb{Z}) \cdot F_r\,.$$

Then D_g^* is clearly the union of all rational boundary components of D_g and is called the *rational closure* of D_g. We let $\Gamma_g := Sp(g,\mathbb{Z})$ for brevity. Then we obtain the so-called Satake-Baily-Borel compactification $A_g^* := \Gamma_g \backslash D_g^*$ of $A_g := \Gamma_g \backslash D_g$. Let F be a *rational* boundary component of D_g. We denote by $P(F)$, $W(F)$, $U(F)$ the parabolic subgroup associated with F, the unipotent radical of $P(F)$ and the center of $W(F)$ respectively. We set $V(F) := W(F)/U(F)$. Since $P(g \cdot F) = gP(F)g^{-1}$ for $g \in Sp(g,\mathbb{R})$, it is enough to investigate the structures of these groups for the standard rational boundary components F_r ($0 \le r \le g$).

Now we take $F = F_r$ for some r with $0 \le r \le g$. We define $D(F) := U(F)_{\mathbb{C}} \cdot D_g \subset \hat{D}_g$. Here $\hat{D}_g := B \backslash Sp(g,\mathbb{R})_{\mathbb{C}}$ is the compact dual of D_g with B a parabolic subgroup of $Sp(g,\mathbb{R})_{\mathbb{C}}$. It is obvious that $U(F)_{\mathbb{C}} \cong S_{g-r}(\mathbb{C})$ and $D(F) \cong F \times V(F) \times U(F)_{\mathbb{C}}$ analytically. We observe that $U(F)$ acts on $D(F)$ as the linear translation on the factor $U(F)_{\mathbb{C}}$. The isomorphism $\varphi : D(F) \longrightarrow F \times V(F) \times U(F)_{\mathbb{C}}$ is given by

$$\varphi\left(\begin{pmatrix} Z_1 & Z_2 \\ * & Z_3 \end{pmatrix} \right) := (Z_1, Z_2, Z_3), \quad Z_1 \in D_r, \ Z_2 \in \mathbb{C}^{(r,g-r)}, \ Z_3 \in S_{g-r}(\mathbb{C}).$$

We define the mapping $\Phi_F : D(F) \longrightarrow U(F)$ by
(4.2)
$$\Phi_F((Z_1, Z_2, Z_3)) := \operatorname{Im} Z_3 - {}^t(\operatorname{Im} Z_2)\,(\operatorname{Im} Z)^{-1}\,(\operatorname{Im} Z_2), \quad (Z_1, Z_2, Z_3) \in D(F).$$

Then $D_g \cong H_g$ is characterized by $\Phi_F(Z) > 0$ for all $Z \in D_g$. This is the realization of a Siegel domain of the third kind. We let $C(F)$ be the cone of real positive symmetric matrices of degree $g - r$ in $U(F) \cong S_{g-r}(\mathbb{R})$. Clearly we have $D_g = \Phi^{-1}(C(F))$. We define

$$G_h(F) := \mathrm{Aut}\,(F) \quad (\text{modulo finite group})$$

and

$$G_l(F) := \mathrm{Aut}\,(U(F),\, C(F)).$$

Then it is easy to see that

$$P(F) \;=\; (\,G_h(F) \times G_l(F)\,) \ltimes W(F) \quad (\text{the semidirect product})$$

We obtain the natural projections $p_h : P(F) \longrightarrow G_h(F)$ and $p_l : P(F) \longrightarrow G_l(F)$.

Step I : Partial compactification for a rational bounadry component.

Now we let Γ be an arithmetic subgroup of $Sp(g, \mathbb{R})$. We let

$$
\begin{aligned}
\Gamma(F) : \quad & = \Gamma \cap P(F), \\
\bar{\Gamma}(F) : \quad & = p_l(\Gamma(F)) \subset G_l(F), \\
U_\Gamma(F) : \quad & = \Gamma \cap U(F), \quad \text{a lattice in } U(F), \\
W_\Gamma(F) : \quad & = \Gamma \cap W(F).
\end{aligned}
$$

We note that $\bar{\Gamma}(F)$ is an arithmetic subgroup of $G_l(F)$.

Let $\Sigma_F = \left\{ \sigma_\alpha^F \right\}$ be a $\bar{\Gamma}(F)$-admissible polyhedral decomposition of $C(F)$. We set $D(F)' := D(F)/U(F)_{\mathbb{C}}$. Since $D(F)' \cong F \times V(F)$, the projection $\pi_F : D(F) \longrightarrow D(F)'$ is a principal $U(F)_{\mathbb{C}}$-bundle over $D(F)'$. The map

(4.3) $\qquad \pi_{F,\Gamma} : U_\Gamma(F) \backslash D(F) \cong F \times V(F) \times (U_\Gamma(F) \backslash U(F)_{\mathbb{C}}) \longrightarrow D(F)'$

is a principal $T(F)$-bundle with the structure group $T(F) := U_\Gamma(F) \backslash U(F)_{\mathbb{C}} \cong (\mathbb{C}^*)^q$, where $q = \dfrac{(g-r)(g-r+1)}{2}$. Let X_{Σ_F} be a normal torus embedding of $T(F)$. We note that X_{Σ_F} is determined by Σ_F. Then we obtain a fibre bundle

(4.4) $\qquad \mathcal{X}(\Sigma_F) := (U_\Gamma(F) \backslash D(F)) \times_{T(F)} \mathrm{X}_{\Sigma_F}$

over $D(F)'$ with fibre X_{Σ_F}. We denote by $\mathbf{X}\,(\Sigma_F)$ the interior of the closure of $U_\Gamma(F) \backslash D_g$ in $\mathcal{X}(\Sigma_F)$ (because $D_g \subset D(F)$). $\mathbf{X}(\Sigma_F)$ has a fibrewise $T(F)$-orbit decomposition $\coprod_\mu O(\mu)$ such that

$$
\begin{aligned}
(i) \quad & \text{each } O(\mu) \text{ is an algebraic torus bundle over } D(F)', \\
(ii) \quad & \sigma_\mu \prec \sigma_\nu \ \text{ iff } \ \overline{O(\mu)} \supseteq O(\nu), \\
(iii) \quad & \dim \sigma_\mu + \dim O(\mu) = \dim D(F), \\
(iv) \quad & \text{for } \sigma_\mu = 0, \ O(\mu) = U_\Gamma(F) \backslash D(F).
\end{aligned}
$$

We define

$$O(F) := \bigcup_{\sigma_\alpha^F \cap C(F) \neq \emptyset} O(\alpha) \subset \mathbf{X}(\Sigma_F)$$

and

$$\bar{O}(F) := \Gamma(F)/U_\Gamma(F)\backslash O(F).$$

We note that $O(F_g) = D_g$ and $\bar{O}(F_g) = \Gamma\backslash D_g$. We set

(4.5) $$\mathbf{Y}(\Sigma_F) := \Gamma(F)/U_\Gamma(F)\backslash \mathbf{X}(\Sigma_F).$$

We note that $\Gamma(F)/U_\Gamma(F)$ acts on $\mathbf{Y}(\Sigma_F)$ properly discontinuously. Then we can show that $\mathbf{Y}(\Sigma_F)$ has a canonical quotient structure of a normal analytic space and $\bar{O}(F)$ is a closed analytic set in $\mathbf{Y}(\Sigma_F)$.

Step II : Gluing.

Let $\Sigma := \{\, \Sigma_F \mid F \text{ is a rational boundary component of } D_g \,\}$ be a Γ-admissible family of polyhedral decompositions. We put

$$\widetilde{(\Gamma\backslash D_g)} := \cup_{F:\text{rational}}\mathbf{X}(\Sigma_F).$$

We define the equivalence relation $\smile$ on $\widetilde{(\Gamma\backslash D_g)}$ as follows:

$$X_1 \smile X_2, \quad X_1 \in \mathbf{X}(\Sigma_{F_1}), \quad X_2 \in \mathbf{X}(\Sigma_{F_2})$$

iff there exist a rational boundary component F, an element $\gamma \in \Gamma$ such that $F_1 \prec F$, $\gamma F_2 \prec F$ and there exists an element $X \in \mathbf{X}(\Sigma_F)$ such that $\pi_{F,F_1}(X) = X_1$, $\pi_{F,F_2}(X) = \gamma X_2$, where

$$\pi_{F,F_1} : \mathbf{X}(\Sigma_F) \longrightarrow \mathbf{X}(\Sigma_{F_1}), \quad \pi_{F,F_2} : \mathbf{X}(\Sigma_F) \longrightarrow \mathbf{X}(\Sigma_{\gamma F_2}).$$

The space $\overline{(\Gamma\backslash D_g)} := \widetilde{(\Gamma\backslash D_g)}/\smile$ is called the *toroidal compactification* of $\Gamma\backslash D_g$ associated with Σ. It is known that $\overline{(\Gamma\backslash D_g)}$ is a Hausdorff analytic variety containing $\Gamma\backslash D_g$ as an open dense subset. For a *neat* arithmetic subgroup Γ, we can obtain a smooth projective toroidal compactification of $\Gamma\backslash D_g$.

II. A toroidal compactification of the universal abelian variety

For a positive integer $g \in \mathbb{Z}^+$, we put $X := \mathbb{Z}^g$. Let $B(X)$ be the $\mathbb{Z}$-module of integral valued symmetric bilinear forms on X and let $B(X)_\mathbb{R} := B(X) \otimes_\mathbb{Z} \mathbb{R}$. Let $C(X) \subset B(X)_\mathbb{R}$ be the convex cone of all positive semi-positive symmetric bilinear forms on $X_\mathbb{R}$ whose radicals are defined over $\mathbb{Q}$. We let X^* be the dual of X. For a positive integer $s \in \mathbb{Z}^+$, we let

$$\tilde{B}_s(X) := B(X) \times (X^*)^s \quad \text{and} \quad \tilde{B}_s(X)_\mathbb{R} := \tilde{B}_s(X) \otimes_\mathbb{Z} \mathbb{R}.$$

Then the semidirect product $GL(X) \ltimes X^s$ acts on $\tilde{B}_s(X)_\mathbb{R}$ in the natural way and the projection $\tilde{B}_s(X)_\mathbb{R} \longrightarrow B(X)_\mathbb{R}$ is equivariant with respect to the canonical morphism $GL(X) \ltimes X^s \longrightarrow GL(X)$. Inside $\tilde{B}_s(X)_\mathbb{R}$ we obtain the cone $\tilde{C}_s(X)$ consisting

of $q = (b; \ell_1, \cdots, \ell_s) \in \tilde{B}_s(X)_{\mathbb{R}}$ such that $b \in C(X)$ and each ℓ_j vanishes on the radical of b.

Let a $GL(X)$-admissible polyhedral cone decomposition $\mathcal{C} = \{\sigma_\alpha\}$ of $C(X)$ be given. A $GL(X) \ltimes X^s$-admissible polyhedral cone decomposition $\tilde{\mathcal{C}} = \{\tau_\beta\}$ of $\tilde{C}_s(X)$ relative to $\mathcal{C} = \{\sigma_\alpha\}$ is defined to be a collection $\tilde{\mathcal{C}} = \{\tau_\beta\}$ such that

(1) each τ_β is a non-degenerate rational polyhedral cone which is open in the smallest $\mathbb{R}$-subspace containing it;

(2) any face of a $\tau_\beta \in \tilde{\mathcal{C}}$ belongs to $\tilde{\mathcal{C}}$;

(3) $\tilde{C}_s(X) = \cup_{\tau_\beta \in \tilde{\mathcal{C}}} \tau_\beta$;

(4) $\tilde{\mathcal{C}}$ is invariant under the action of $GL(X) \ltimes X^s$ and there are only finitely many $GL(X) \ltimes X^s$-orbits;

(5) any $\tau_\beta \in \tilde{\mathcal{C}}$ maps into a $\sigma_\alpha \in \mathcal{C}$ under the natural projection $\tilde{C}_s(X) \longrightarrow C(X)$.

We call $\tilde{\mathcal{C}}$ *equidimensional* if in (5) of the above definition each $\tau_\beta \in \tilde{\mathcal{C}}$ maps onto a $\sigma_\alpha \in \mathcal{C}$. Again, $\tilde{\mathcal{C}}$ is called *smooth* or *regular* if each $\tau_\beta \in \tilde{\mathcal{C}}$ is generated by part of a $\mathbb{Z}$-basis of $\tilde{B}_s(X)$. According to the reduction theory [3], there exists a smooth equidimensional $GL(X) \ltimes X^s$-admissible polyhedral cone decomposition $\tilde{\mathcal{C}}$ of $\tilde{C}_s(X)$ relative to $\mathcal{C}$. Let F be the split torus $\tilde{B}_s(X)_{\mathbb{R}} \otimes_{\mathbb{Z}} G_{\mathbf{m}}$. The choice of a polyhedral cone decomposition $\tilde{\mathcal{C}} = \{\tau_\beta\}$ of $\tilde{C}_s(X)$ as above provides us with a torus embedding $F \hookrightarrow \bar{F}$. Then $\bar{F}$ is stratified by F-orbits and $GL(X) \ltimes X^s$ acts on $\bar{F}$ preserving this stratification. Therefore we obtain the toroidal compactification $\bar{A}_{g,s}$ of the universal abelian variety $A_{g,s} := \Gamma_{g,s}^J \backslash H_g \times \mathbb{C}^{(s,g)}$ with $\Gamma_{g,s}^J := \Gamma_g \ltimes H_{\mathbb{Z}}^{(g,h)}$. We collect some properties of the toroidal compactification $\bar{A}_{g,s}$.

(a) $\bar{A}_{g,s}$ is a Haudorff analytic variety containing $A_{g,s}$ as an open dense subset.

(b) $\bar{A}_{g,s}$ has a stratification parametrized by the $GL(X) \ltimes X^s$-orbits of cones $\tau_\beta \in \tilde{\mathcal{C}}$.

(c) The toroidal compactification $\bar{A}_{g,s}$ depends on the choice of a smooth equidimensional $GL(X) \ltimes X^s$-admissible polyhedral cone decomposition $\tilde{\mathcal{C}} = \{\tau_\beta\}$ of $\tilde{C}_s(X)$ relative to $\mathcal{C}$. In order to indicate this dependence we write $\bar{A}_{g,s}(\tilde{\mathcal{C}})$ instead of $\bar{A}_{g,s}$. The natural projection $\pi : A_{g,s} \longrightarrow A_g$ extends to a proper morphism $\bar{\psi} : \bar{A}_{g,s} \longrightarrow \bar{A}_g$.

Now we recall [22], p. 197 that an *admissible homogeneous principal polarization function* of $\{\tau_\beta\} \longrightarrow \{\sigma_\alpha\}$ is a piecewise linear function $\tilde{\phi} : \tilde{C}_s(X) \longrightarrow \mathbb{R}$ satisfying the following conditions

(P1) $\tilde{\phi}$ is continuous and $GL(X)$-invariant;

(P2) $\tilde{\phi}$ takes rational values on $\tilde{B}_s(X) \cap \tilde{C}_s(X)$ with bounded denominators;

(P3) $\tilde{\phi}$ is homogeneous, i.e., $\tilde{\phi}(t \cdot q) = t \cdot \tilde{\phi}(q)$ for all real $t \geq 0$ and all $q \in \tilde{C}_s(X)$;

(P4) $\tilde{\phi}$ is linear on each $\tau_\beta \in \tilde{\mathcal{C}}$;

(P5) $\tilde{\phi}$ is convex in the sense that

$$\tilde{\phi}(t \cdot q + (1-t) \cdot q') \geq t \cdot \tilde{\phi}(q) + (1-t) \cdot \tilde{\phi}(q')$$

for all $t \in \mathbb{R}$ with $0 \leq t \leq 1$ and any $q, q' \in \tilde{C}_s(X)$.

(P6) $\tilde{\phi}$ is strictly convex, that is, for each $\sigma_\alpha \in \mathcal{C} = \{\sigma_\alpha\}$ and each $\tau_\beta \in \tilde{\mathcal{C}} = \{\tau_\beta\}$ lying over σ_α, there exist a finite number of linear functionals $\ell_i : \tilde{B}_s(X) \longrightarrow \mathbb{R}$, $1 \leq i \leq m$ with $\ell_i \geq \tilde{\phi}$ on the preimage of σ_α for each i and

$$\tau_\beta = \{\, q \in \tilde{C}_s(X) \,|\, q \text{ lies over } \sigma_\alpha \text{ and } \tilde{\phi}(q) = \ell_i(q) \text{ for each } i \,\}.$$

(P7) There exists a rational positive number r such that for each $\mu = (\mu_1, \cdots, \mu_s) \in X^s$, the function

$$\tilde{\phi} - \tilde{\phi} \circ T_\mu \,:\, q \longmapsto f(q) - f(\mu \cdot q)$$

is equal to r times (restriction to $\tilde{C}_s(X)$ of) the linear functional $\tilde{\chi}_\mu$ on $\tilde{B}_s(X)$, where for $q = (b; \ell_1, \cdots, \ell_s) \in \tilde{C}_s(X)$,

$$\tilde{\chi}_\mu(q) := \sum_{1 \leq i \leq s} a_i(\mu_i) = \sum_{1 \leq i \leq s} \{\, b(\mu_i, \mu_i) + 2 \cdot \ell_i(\mu_i)\,\}.$$

The conditions (P1)-(P7) above constitute a kind of convexity conditions on $\{\tau_\beta\} \longrightarrow \{\sigma_\alpha\}$. They imply that the morphism $\bar{A}_{g,s} \longrightarrow \bar{A}_g$ attached to $\{\tau_\beta\} \longrightarrow \{\sigma_\alpha\}$ is *projective*. Indeed, the theory of torus embeddings shows that an admissible homogeneous principal polarization function $\tilde{\phi} : \tilde{C}_s(X) \longrightarrow \mathbb{R}$ gives rise to an invertible sheaf $\bar{\mathcal{L}}(\tilde{\phi})$, which is ample on $\bar{A}_{g,s}(\tilde{\mathcal{C}})$ relative to $\bar{A}_g(\mathcal{C})$.

5. The Automorphic Vector Bundle $E_{\rho,\mathcal{M}}$

Let ρ and $\mathcal{M}$ be as before in section 2. Assume that Γ is a subgroup of $\Gamma_g := Sp(g, Z)$ of finte index which acts freely on H_g and $-E_{2g} \notin \Gamma$. Then $\Gamma^J := \Gamma \ltimes H_{\mathbb{Z}}^{(g,h)}$ acts on $H_{g,h} := H_g \times \mathbb{C}^{(h,g)}$ properly discontinuously. We consider the automorphic factor $J_{\mathcal{M},\rho} : G_{g,h}^J \times H_{g,h} \longrightarrow GL(V_\rho)$ defined by

$$
\begin{aligned}
J_{\mathcal{M},\rho}(\tilde{g}, (Z, W)) := \quad & e^{2\pi i \sigma(\mathcal{M}[W + \lambda Z + \mu](CZ+D)^{-1}C)} \\
& \times e^{-2\pi i \sigma(\mathcal{M}(\lambda Z\,^t\lambda + 2\lambda^t W + \kappa + \mu\,^t\lambda))} \rho(CZ + D),
\end{aligned}
$$

where $\tilde{g} = (M, (\lambda, \mu, \kappa)) \in G^J$ with $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g, \mathbb{R})$. Then $J_{\mathcal{M},\rho}$ defines the automorphic vector bundle $E_{\rho,\mathcal{M}} := H_{g,h} \times_{\Gamma^J} V_\rho$ over $A_{g,h,\Gamma} := \Gamma^J \backslash H_{g,h}$. By the definition, Jacobi forms in $J_{\rho,\mathcal{M}}(\Gamma)$ may be considered as holomorphic sections of the vector bundle $E_{\rho,\mathcal{M}}$ with some additional cusp condition. For $g \geq 2$, this additional condition may be dropped according to Köcher principle. Let $\bar{A}_{g,h,\Gamma}$ be a toroidal compactification given by a regular Γ-admissible family Σ of polyhedral decompositions.

Without proof we provides our results.

Theorem 5.1. $A_{g,h,\Gamma}$ *is contained in* $\bar{A}_{g,h,\Gamma}$ *as a Zariski open subset.* $E_{\rho,\mathcal{M}}$ *can be extended uniquely to the holomorphic vector bundle* $\bar{E}_{\rho,\mathcal{M}}$ *over* $\bar{A}_{g,h,\mathcal{M}}$. *And* $H^i(A_{g,h,\Gamma}, E_{\rho,\mathcal{M}})$
$\cong H^i(\bar{A}_{g,h,\mathcal{M}}, \bar{E}_{\rho,\mathcal{M}})$. *In particular, the dimension of* $J_{\rho,\mathcal{M}}$ *is finite dimensional.*
vspace0.1in
Definition 5.2. Let ρ be an irreducible rational representation of $GL(g, \mathbb{C})$ with its highest weight $(\lambda_1, \lambda_2, \cdots, \lambda_g)$. We call the number of j ($1 \leq j \leq g$) such that $\lambda_j = \lambda_g$ the *corank* of ρ which is denoted by corank (ρ). The number $k(\rho) := \lambda_g$ is called the *weight* of ρ.

Theorem 5.3. *Let* $2\mathcal{M}$ *be an even unimodular positive definite matrix of degree* h. *Let* ρ *be an irreducible finite dimensional representation of* $GL(g, \mathbb{C})$ *with highest weight* $\rho = (\lambda_1, \cdots, \lambda_g)$. *Let* $\lambda(\rho)$ *be the number of* $\lambda_i's$ *such that* $\lambda_i = k(\rho) + 1 = \lambda_g + 1$, $1 \leq i \leq g$. *Assume that* ρ *satisfies the following conditions :*

$$[a] \quad \rho(A) = \rho(-A) \quad \text{for all } A \in GL(g, \mathbb{C}),$$
$$[b] \quad \lambda(\rho) < 2(g - k(\rho) - \text{corank}\,(\rho)\,) + h.$$

Then $H^0(A_{g,h,\Gamma}, E_{\rho,\mathcal{M}}) = 0$.

Proof. The proof can be found in [80].

Corollary 5.4. *Let* $2\mathcal{M}$ *be as above in Theorem 5.3. Assume that* $2k(\rho) \leq g + h - 2\text{corank}\,(\rho)$. *Then* $H^0(A_{g,h,\Gamma}, E_{\rho,\mathcal{M}}) = 0$.

Remark 5.5. N.-P. Skoruppa [Sk] proved that $J_{1,m}(\Gamma_1) = 0$ for any nonnegative integer m. It is interesting to give the geometric proofs of this fact and Theorem 5.3.

We give the following open problems :

Problem 1. Give the explicit dimension formula or estimate for $H^0(A_{g,h,\Gamma}, E_{\rho,\mathcal{M}})$.

Problem 2. Compute the cohomology groups $H^k(A_{g,h,\Gamma}, E_{\rho,\mathcal{M}})$ explicitly. Here $0 \leq k \leq \dfrac{g(g + 2h + 1)}{2}$.

Problem 3. Under which conditions is $E_{\rho,\mathcal{M}}$ *ample*?

Problem 4. Discuss the analogue of Hirzebruch's proportionality theorem for $E_{\rho,\mathcal{M}}$(cf. [45]).

6. Smooth Compactification of Siegel Moduli Spaces and Open Problems

Let $\Gamma_g(k)$ be the principal congruence subgroup of $Sp(g, Z)$ of level k and let H_g be the Siegel upper-half plane of degree g. We assume that $k \geq 3$. This implies

that $\Gamma_g(k)$ is a *neat* arithmetic subgroup. Let $\bar{X}$ be the toroidal compactification of $X := \Gamma_g(k)\backslash H_g$ from $\Gamma_g(k)$-admissible family given by the central cone decomposition $\sum_{cent}$ or a refinement of $\sum_{cent}$. Then the boundary $D := \bar{X} - X = \sum_{i=1}^m D_i$ is a divisor of $\bar{X}$ with normal crossing, that is, each D_i is an irreducible smooth divisor of $\bar{X}$ and $D_1, \cdots, D_m$ intersect transversally. If $g \leq 4$, we have the following results obtained by Wang [63].

Theorem 6.1. *(1) Each divisor is algebraically isomorphic to*

$$\bar{Y}_{g-1} := \overline{\Gamma_{g-1}^J(k)\backslash(H_{g-1} \times C^{g-1})}.$$

Here $\Gamma_{g-1}^J(k) := \Gamma_{g-1}(k) \ltimes (kZ)^{g-1}$ is the Jacobi modular group acting on the homogeneous space $W_{g-1} := H_{g-1} \times C^{g-1}$ in a usual way and $\bar{Y}_{g-1}$ is the compactification of the universal family $Y_{g-1} := \Gamma_{g-1}^J(k)\backslash(H_{g-1} \times C^{g-1})$ of abelian varieties induced from the same $\Gamma_g(k)$-admissible family.

(2) All D_i intersect along the boundary $\bar{Y}_{g-1} - Y_{g-1}$.

We have several *natural* questions.

Problem 6.2. Describe $\bar{Y}_{g-1}$ and $\bar{Y}_{g-1} - Y_{g-1}$ explicitly in terms of Jacobi forms. More generally, *describe $\bar{Y}_r$ and $\bar{Y}_r - Y_r$ when $Y_r := \Gamma_r(k) \ltimes H_Z^{(r,k)}\backslash H_r \times C^{(r,k)}$ ($1 \leq r \leq g$).*

Problem 6.3. Describe the field of meromorphic functions on $\bar{Y}_{g-1}$ or $\bar{Y}_r$.

Problem 6.4. Can any $\Gamma_{g-1}^J(k)$-invariant or $\Gamma_r^J(k)$-invariant meromorphic function on Y_{g-1} or Y_r be expressed by a quotient of two Jacobi forms of the same weight and index?

7. The Boundary of the Satake Compactification

Let Γ be a discrete subgroup of $Sp(g, \mathbb{Q})$ which is commensurable with Γ_g. We denote by $M_k(\Gamma)$ the complex vector space consisting of Siegel modular forms of weight k with respect to Γ ($k \in \mathbb{Z}$). These vector spaces generate a positively graded ring

$$M(\Gamma) := \oplus_{k \geq 0} M_k(\Gamma)$$

which are integrally closed and of finite type over $M_0(\Gamma) = \mathbb{C}$. The projective variety $A_{g,\Gamma}^*$ associated with $M(\Gamma)$ contains a Zariski open subset which is complex analytically isomorphic to $A_{g,\Gamma} := \Gamma\backslash H_g$. In addition, the boundary $\partial A_{g,\Gamma}^* := A_{g,\Gamma}^* - A_{g,\Gamma}$ is a disjoint union of a finite number of rational boundary components of H_g.

From now on, we let $\Gamma := \Gamma_g(k)$ be the principal congruence subgroup of Γ_g of level k. We write $g = p + q$ for $0 \leq p < g$. We write an element Z of H_g as

$$\begin{pmatrix} \tau & W \\ {}^tW & T \end{pmatrix}, \quad \tau \in H_p, \ W \in \mathbb{C}^{(p,q)}, \ T \in H_q,$$

or simply $Z = (\tau, W, T)$. The Siegel operator $\Phi : M(\Gamma_g(k)) \longrightarrow M(\Gamma_p(k))$ defined by

$$(7.1) \qquad (\Phi f)(\tau) := \lim_{\text{Im } T \to 0} f\left(\begin{pmatrix} \tau & W \\ * & T \end{pmatrix}\right) = \lim_{c \to 0} f\left(\begin{pmatrix} \tau & 0 \\ 0 & icE_q \end{pmatrix}\right)$$

is a weight-preserving homomorphism which is almost surjective in the sense that it is surjective for all large weights. Thus we have a canonical holomorphic embedding $\Phi^* : A^*_{p, \Gamma_p(k)} \longrightarrow A^*_{g, \Gamma_g(k)}$. We can see that the image of $A_{p, \Gamma_p(k)} = \Gamma_p(k) \backslash H_p$ is a quasi-projective subvariety of $A^*_{g, \Gamma_g(k)}$ and that $Sp(g, \mathbb{Z}/k\mathbb{Z})$ acts on $A^*_{g, \Gamma_g(k)}$ as automorphisms. $Sp(g, \mathbb{Z}/k\mathbb{Z})$ transforms $\Phi^*(A_{p, \Gamma_p(k)})$ to its conjugates. Thus we have

$$\begin{aligned} \partial A^*_{g, \Gamma_g(k)} : \quad &= A^*_{g, \Gamma_g(k)} - A_{g, \Gamma_g(k)} \\ &= \bigcup_{\gamma \in Sp(g, \mathbb{Z}/k\mathbb{Z})} \coprod_{l=0}^{g-1} \gamma \cdot \Phi^*(A_{l, \Gamma_l(k)}) \end{aligned}$$

So in order to investigate the boundary $\partial A^*_{g, \Gamma_g(k)}$, it is enough to investigate the boundary points in the image $\Phi^*(A_{p, \Gamma_p(k)})$ of $A_{p, \Gamma_p(k)} = \Gamma_p(k) \backslash H_p$ under Φ^* for $0 \le p < g$.

Omitting the detail, we state the following results.

Theorem 7.1(Igusa). *Let τ_0 be an element of H_p. Then the analytic local ring $\mathcal{O}$ of $A^*_{g, \Gamma_g(k)}$ at the image point of τ_0 under Φ^* consists of convergent series of the following form*

$$f(\tau, W, T) = \sum_{\mathcal{M}} \left(\sum_u \phi_{\mathcal{M}}(\tau, W\,{}^t u)\, e^{\frac{2\pi i \sigma(\mathcal{M}[u]T)}{k}} \right), \quad \phi_{\mathcal{M}} \in J_{0, \mathcal{M}}(\Gamma_g(k)),$$

where $\mathcal{M}$ runs over the equivalent classes of inequivalent half-integral semi-positive symmetric matrices of degree q, $\phi_{\mathcal{M}}$ is a holomorphic function defined on $V \times \mathbb{C}^{(q,p)}$ for some open neighborhood V of τ_0 in H_p and u runs over distinct $\mathcal{M}[u]$ for $u \in GL(q, \mathbb{Z})(k)$.

Theorem 7.2(Igusa). *The ideal I in $\mathcal{O}$ associated with the boundary $\partial A^*_{g, \Gamma_g(k)} = A^*_{g, \Gamma_g(k)} - A_{g, \Gamma_g(k)}$ consists of convergent series*

$$\sum_{\mathcal{M}} \left(\sum_u \phi_{\mathcal{M}}(\tau, W\,{}^t u)\, e^{\frac{2\pi i \sigma(\mathcal{M}[u]T)}{k}} \right), \quad \phi_{\mathcal{M}} \in J_{0, \mathcal{M}}(\Gamma_g(k)),$$

where $\mathcal{M}$ runs over inequivalent symmetric positive definite half-integral matrices of degree q, $\phi_{\mathcal{M}}$ is a holomorphic function defined on $V \times \mathbb{C}^{(q,p)}$ for some open neighborhood V of τ_0 in H_p and u runs over distinct $\mathcal{M}[u]$ for $u \in GL(q, \mathbb{Z})(k)$.

8. Singular Jacobi Forms

In this section, we discuss the notion of singular Jacobi forms. Without loss of

generality we may assume that $\mathcal{M}$ is positive definite. For simplicity, we consider the case that Γ is the Siegel modular group Γ_g of degree g.

Let g and h be two positive integers. We recall that $\mathcal{M}$ is a symmetric positive definite, half-integral matrix of degree h. We let

$$\mathcal{P}_g := \{Y \in \mathbb{R}^{(g,g)} \,|\, Y = {}^tY > 0\,\}$$

be the open convex cone of positive definite matrices of degree g in the Euclidean space $\mathbb{R}^{\frac{g(g+1)}{2}}$. We define the differential operator $M_{g,h,\mathcal{M}}$ on $\mathcal{P}_g \times \mathbb{R}^{(h,g)}$ defined by

$$M_{g,h,\mathcal{M}} := \det(Y) \cdot \det\left(\frac{\partial}{\partial Y} + \frac{1}{8\pi} {}^t\left(\frac{\partial}{\partial V}\right) \mathcal{M}^{-1}\left(\frac{\partial}{\partial V}\right)\right),$$

where

$$Y = (y_{\mu\nu}) \in \mathcal{P}_g, \quad V = (v_{kl}) \in \mathbb{R}^{(h,g)}, \quad \frac{\partial}{\partial Y} = \left(\frac{1 + \delta_{\mu\nu}}{2} \frac{\partial}{\partial y_{\mu\nu}}\right)$$

and

$$\frac{\partial}{\partial V} = \left(\frac{\partial}{\partial v_{kl}}\right).$$

Yang [85] characterized singular Jacobi forms as follows:

Theorem 8.1. *Let $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$ be a Jacobi form of index $\mathcal{M}$ with respect to a finite dimensional rational representation ρ of $GL(g, \mathbb{C})$. Then the following conditions are equivalent :*
(1) f is a singular Jacobi form.
(2) f satisfies the differential equation $M_{g,h,\mathcal{M}} f = 0$.

Theorem 8.2. *Let ρ be an irreducible finite dimensional representation of $GL(g, \mathbb{C})$. Then there exists a nonvanishing singular Jacobi form in $J_{\rho,\mathcal{M}}(\Gamma_g)$ if and only if $2k(\rho) < g + h$. Here $k(\rho)$ denotes the weight of ρ.*

For the proofs of the above theorems we refer to [85], Theorem 4.1 and Theorem 4.5.

Exercise 8.3. Compute the eigenfunctions and the eigenvalues of $M_{g,h,\mathcal{M}}$(cf. [85], pp. 2048-2049).

Now we consider the following group $GL(g, \mathbb{R}) \ltimes H_{\mathbb{R}}^{(g,h)}$ equipped with the multiplication law

$$(A, (\lambda, \mu, \kappa)) * (B, (\lambda', \mu', \kappa'))$$
$$= (AB, (\lambda B + \lambda', \mu\, {}^tB^{-1} + \mu', \kappa + \kappa' + \lambda B\, {}^t\mu' - \mu\, {}^tB^{-1}\, {}^t\lambda')),$$

where $A, B \in GL(g, \mathbb{R})$ and $(\lambda, \mu, \kappa), (\lambda', \mu', \kappa') \in H_{\mathbb{R}}^{(g,h)}$. We observe that $GL(g, \mathbb{R})$ acts on $H_{\mathbb{R}}^{(g,h)}$ on the right as automorphisms. And we have the canonical action of

$GL(g, \mathbb{R}) \ltimes H_{\mathbb{R}}^{(g,h)}$ on $\mathcal{P}_g \times \mathbb{R}^{(h,g)}$ defined by

$$(8.1) \qquad (A, (\lambda, \mu, \kappa)) \circ (Y, V) := (AY\, {}^tA, (V + \lambda Y + \mu)\, {}^tA),$$

where $A \in GL(g, \mathbb{R}), (\lambda, \mu, \kappa) \in H_{\mathbb{R}}^{(g,h)}$ and $(Y, V) \in \mathcal{P}_g \times \mathbb{R}^{(h,g)}$.

Lemma 8.4. $M_{g,h,\mathcal{M}}$ is invariant under the action of $GL(g, \mathbb{R}) \ltimes \left\{ (0, \mu, 0) \,|\, mu \in \mathbb{R}^{(h,g)} \right\}$.

Proof. It follows immediately from the direct calculation.

We have the following natural questions.

Problem 8.5. Develope the invariant theory for the action of $GL(g, \mathbb{R}) \ltimes H_{\mathbb{R}}^{(g,h)}$ on $\mathcal{P}_g \times \mathbb{R}^{(h,g)}$.

Problem 8.6. Discuss the application of the theory of singular Jacobi forms to the geometry of the universal abelian variety as that of singular modular forms to the geometry of the Siegel modular variety (see Appendix B).

9. The Siegel-Jacobi Operator

Let ρ and $\mathcal{M}$ be the same as in the previous sections. For positive integers r and g with $r < g$, we let $\rho^{(r)} : GL(r, \mathbb{C}) \longrightarrow GL(V_\rho)$ be a rational representation of $GL(r, \mathbb{C})$ defined by

$$\rho^{(r)}(a)v := \rho\left(\begin{pmatrix} a & 0 \\ 0 & E_{g-r} \end{pmatrix} \right) v, \quad a \in GL(r, \mathbb{C}), \quad v \in V_\rho.$$

The Siegel-Jacobi operator $\Psi_{g,r} : J_{\rho,\mathcal{M}}(\Gamma_g) \longrightarrow J_{\rho^{(r)},\mathcal{M}}(\Gamma_r)$ is defined by

$$(9.1) \qquad (\Psi_{g,r}f)(Z, W) := \lim_{t \to \infty} f\left(\begin{pmatrix} Z & 0 \\ 0 & itE_{g-r} \end{pmatrix}, (W, 0) \right),$$

where $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$, $Z \in H_r$ and $W \in \mathbb{C}^{(h,r)}$. It is easy to check that the above limit always exists and the Siegel-Jacobi operator is a linear mapping. Let $V_\rho^{(r)}$ be the subspace of V_ρ spanned by the values $\left\{ (\Psi_{g,r}f)(Z, W) \,|\, f \in J_{\rho,\mathcal{M}}(\Gamma_g), (Z, W) \in H_r \times \mathbb{C}^{(h,r)} \right\}$. Then $V_\rho^{(r)}$ is invariant under the action of the group

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & E_{g-r} \end{pmatrix} : a \in GL(r, \mathbb{C}) \right\} \cong GL(r, \mathbb{C}).$$

We can show that if $V_\rho^{(r)} \neq 0$ and (ρ, V_ρ) is irreducible, then $(\rho^{(r)}, V_\rho^{(r)})$ is also irreducible.

Theorem 9.1. *The action of the Siegel-Jacobi operator is compatible with that of that of the Hecke operator.*

We refer to [83] for a precise detail on the Hecke operators and the proof of the

above theorem.

Problem 9.2. Discuss the injectivity, surjectivity and bijectivity of the Siegel-Jacobi operator.

This problem was partially discussed by Yang [83] and Kramer [40] in the special cases. For instance, Kramer [40] showed that if g is arbitrary, $h = 1$ and $\rho :$ $GL(g, \mathbb{C}) \longrightarrow \mathbb{C}^{\times}$ is a one-dimensional representation of $GL(g, \mathbb{C})$ defined by $\rho(a) :=$ $(\det(a))^k$ for some $k \in \mathbb{Z}^+$, then the Siegel-Jacobi operator

$$\Psi_{g,g-1} : J_{k,m}(\Gamma_g) \longrightarrow J_{k,m}(\Gamma_{g-1})$$

is surjective for $k \gg m \gg 0$.

Theorem 9.3. *Let* $1 \leq r \leq g - 1$ *and let* ρ *be an irreducible finite dimensional representation of* $GL(g, \mathbb{C})$. *Assume that* $k(\rho) > g + r + rank(\mathcal{M}) + 1$ *and that* k *is even. Then*

$$J^{cusp}_{\rho^{(r)}, \mathcal{M}}(\Gamma_r) \subset \Psi_{g,r}(J_{\rho, \mathcal{M}}(\Gamma_g)).$$

Here $J^{cusp}_{\rho^{(r)}, \mathcal{M}}(\Gamma_r)$ *denotes the subspace consisting of all cuspidal Jacobi forms in* $J_{\rho^{(r)}, \mathcal{M}}(\Gamma_r)$.

Idea of Proof. For each $f \in J^{\text{cusp}}_{\rho^{(r)}, \mathcal{M}}(\Gamma_r)$, we can show by a direct computation that

$$\Psi_{g,r}(E^{(g)}_{\rho, \mathcal{M}}(Z, W; f)) = f,$$

where $E^{(g)}_{\rho, \mathcal{M}}(Z, W; f)$ is the Eisenstein series of Klingen's type associated with a cusp form f. For a precise detail, we refer to [94].

Remark 9.4. Dulinski [17] decomposed the vector space $J_{k, \mathcal{M}}(\Gamma_g) (k \in \mathbb{Z}^+)$ into a direct sum of certain subspaces by calculating the action of the Siegel-Jacobi operator on Eisenstein series of Klingen's type explicitly.

For two positive integers r and g with $r \leq g - 1$, we consider the bigraded ring

$$J^{(r)}_{*,*}(\ell) := \oplus_{k=0}^{\infty} \oplus_{\mathcal{M}} J_{k, \mathcal{M}}(\Gamma_r(\ell))$$

and

$$M^{(r)}_*(\ell) := \oplus_{k=0}^{\infty} J_{k,0}(\Gamma_r(\ell)) = \oplus_{k=0}^{\infty}[\Gamma_r(\ell), k],$$

where $\Gamma_r(\ell)$ denotes the principal congruence subgroup of Γ_r of level ℓ and $\mathcal{M}$ runs over the set of all symmetric semi-positive half-integral matrices of degree h. Let

$$\Psi_{r,r-1,\ell} : J_{k, \mathcal{M}}(\Gamma_r(\ell)) \longrightarrow J_{k, \mathcal{M}}(\Gamma_{r-1}(\ell))$$

be the Siegel-Jacobi operator defined by (9.1).

Problem 9.5. Investigate $\mathrm{Proj}\, J^{(r)}_{*,*}(\ell)$ over $M^{(r)}_*(\ell)$ and the quotient space

$$Y_r(\ell) := (\Gamma_r(\ell) \ltimes (\ell\mathbb{Z})^2) \backslash (H_r \ltimes \mathbb{C}^r)$$

for $1 \leq r \leq g - 1$.

The difficulty to this problem comes from the following facts (A) and (B) :
- (A) $J_{*,*}^{(r)}(\ell)$ is not finitely generated over $M_*^{(r)}(\ell)$.
- (B) $J_{k,\mathcal{M}}^{\mathrm{cusp}}(\Gamma_r(\ell)) \neq \ker \Psi_{r,r-1,\ell}$ in general.

These are the facts different from the theory of Siegel modular forms. We remark that Runge([54], pp. 190-194) discussed some parts about the above problem.

10. Invariant Metrics on the Siegel-Jacobi Space

For a brevity, we write $H_{g,h} := H_g \times \mathbb{C}^{(h,g)}$. For a coordinate $(Z, W) \in H_{g,h}$ with $Z = (z_{\mu\nu}) \in H_g$ and $W = (w_{kl}) \in \mathbb{C}^{(h,g)}$, we put

$$
\begin{aligned}
Z &= X + iY, & X &= (x_{\mu\nu}), & Y &= (y_{\mu\nu}) \text{ real,} \\
W &= U + iV, & U &= (u_{kl}), & V &= (v_{kl}) \text{ real,} \\
dZ &= (dz_{\mu\nu}), & dX &= (dx_{\mu\nu}), & dY &= (dy_{\mu\nu}), \\
dW &= (dw_{kl}), & dU &= (du_{kl}), & dV &= (dv_{kl}),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial Z} &= \left(\frac{1+\delta_{\mu\nu}}{2} \frac{\partial}{\partial z_{\mu\nu}} \right), & \frac{\partial}{\partial \overline{Z}} &= \left(\frac{1+\delta_{\mu\nu}}{2} \frac{\partial}{\partial \overline{z}_{\mu\nu}} \right), \\
\frac{\partial}{\partial X} &= \left(\frac{1+\delta_{\mu\nu}}{2} \frac{\partial}{\partial x_{\mu\nu}} \right), & \frac{\partial}{\partial Y} &= \left(\frac{1+\delta_{\mu\nu}}{2} \frac{\partial}{\partial y_{\mu\nu}} \right),
\end{aligned}
$$

$$
\frac{\partial}{\partial W} := \begin{pmatrix} \frac{\partial}{\partial w_{11}} & \cdots & \frac{\partial}{\partial w_{h1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial w_{1g}} & \cdots & \frac{\partial}{\partial w_{hg}} \end{pmatrix}, \quad \frac{\partial}{\partial \overline{W}} := \begin{pmatrix} \frac{\partial}{\partial \overline{w}_{11}} & \cdots & \frac{\partial}{\partial \overline{w}_{h1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial \overline{w}_{1g}} & \cdots & \frac{\partial}{\partial \overline{w}_{hg}} \end{pmatrix},
$$

$$
\frac{\partial}{\partial U} := \begin{pmatrix} \frac{\partial}{\partial u_{11}} & \cdots & \frac{\partial}{\partial u_{h1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial u_{1g}} & \cdots & \frac{\partial}{\partial u_{hg}} \end{pmatrix}, \quad \frac{\partial}{\partial V} := \begin{pmatrix} \frac{\partial}{\partial v_{11}} & \cdots & \frac{\partial}{\partial v_{h1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial v_{1g}} & \cdots & \frac{\partial}{\partial v_{hg}} \end{pmatrix}.
$$

We let

$$
T_g := \left\{ z \in \mathbb{C}^{(g,g)} \mid z = {}^t z \right\}
$$

be the vector space of all $g \times g$ complex *symmetric* matrices. The unitary group $K := U(g)$ of degree g acts on the complex vector space $T_g \times \mathbb{C}^{(h,g)}$ by

$$
(10.1) \qquad k \cdot (z, w) := (k z \, {}^t k, w \, {}^t k), \qquad k \in U(g), \ z \in T_g, \ w \in \mathbb{C}^{(h,g)}.
$$

Then this action induces naturally the action ρ of $U(g)$ on the polynomial algebra $\mathrm{Pol}_{h,g} := \mathrm{Pol}\,(T_g \times \mathbb{C}^{(h,g)})$. We denote by $\mathrm{Pol}_{h,g}^K$ the subalgebra of $\mathrm{Pol}_{h,g}$ consisting of all K-invariants of the action ρ of $K := U(g)$. We also denote by $\mathbb{D}(H_{g,h})$ the algebra of all differential operators on $H_{g,h}$ which is invariant under the action (2.1) of the Jacobi group $G_{g,h}^J$. Then we can show that there exists a natural linear bijection

$$
(10.2) \qquad \Phi : \mathrm{Pol}_{h,g}^K \longrightarrow \mathbb{D}(H_{g,h})
$$

of $\mathrm{Pol}^K_{h,g}$ onto $\mathbb{D}(H_{g,h})$.

Theorem 10.1. *The algebra $\mathbb{D}(H_{g,h})$ is generated by the images under the mapping Φ of the following invariants*

(I1) $p_j(z,w) := \sigma((z\bar{z})^j), \qquad 1 \leq j \leq g,$

(I2) $\psi^{(1)}_k(z,w) := (w\,{}^t\bar{w})_{kk}, \qquad 1 \leq k \leq h,$

(I3) $\psi^{(2)}_{kp}(z,w) := \mathrm{Re}(w\,{}^t\bar{w})_{kp}, \qquad 1 \leq k < p \leq h,$

(I4) $\psi^{(2)}_{kp}(z,w) := \mathrm{Im}(w\,{}^t\bar{w})_{kp}, \qquad 1 \leq k < p \leq h,$

(I5) $f^{(1)}_{kp}(z,w) := \mathrm{Re}(w\bar{z}\,{}^t w)_{kp}, \qquad 1 \leq k \leq p \leq h$

and

(I6) $f^{(2)}_{kp}(z,w) := \mathrm{Im}(w\bar{z}\,{}^t w)_{kp}, \qquad 1 \leq k \leq p \leq h.$

In particular, $\mathbb{D}(H_{g,h})$ is not commutative.

Theorem 10.1'. *The algebra $\mathbb{D}(H_{1,1})$ is generated by the following differential operators*

$$D := \quad y^2\left(\tfrac{\partial^2}{\partial x^2} + \tfrac{\partial^2}{\partial y^2}\right) + v^2\left(\tfrac{\partial^2}{\partial u^2} + \tfrac{\partial^2}{\partial v^2}\right)$$
$$+ 2\,y\,v\left(\tfrac{\partial^2}{\partial x\partial u} + \tfrac{\partial^2}{\partial y\partial v}\right),$$

$$\Psi := y\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right),$$

$$D_1 := 2\,y^2\,\frac{\partial^3}{\partial x\partial u\partial v} - y^2\,\frac{\partial}{\partial y}\left(\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2}\right) + \left(v\,\frac{\partial}{\partial v} + 1\right)\Psi$$

and

$$D_2 := y^2\,\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial v^2} - \frac{\partial^2}{\partial u^2}\right) - 2\,y^2\,\frac{\partial^3}{\partial y\partial u\partial v} - v\,\frac{\partial}{\partial u}\Psi,$$

where $\tau = x + iy$ and $z = u + iv$ with real variables x, y, u, v. Moreover, we have

$$[D,\ \Psi] := D\Psi - \Psi D = \quad 2\,y^2\,\tfrac{\partial}{\partial y}\left(\tfrac{\partial^2}{\partial u^2} - \tfrac{\partial^2}{\partial v^2}\right) - 4\,y^2\,\tfrac{\partial^3}{\partial x\partial u\partial v}$$
$$- 2\left(v\,\tfrac{\partial}{\partial v}\Psi + \Psi\right).$$

In particular, the algebra $\mathbb{D}(H_{1,1})$ is not commutative.

Theorem 10.2. *The following metric*

(10.3)
$$ds^2_{g,h} := \quad \sigma\left(Y^{-1}dZ\,Y^{-1}d\overline{Z}\right) + \sigma\left(Y^{-1}\,{}^tV\,V\,Y^{-1}dZ\,Y^{-1}d\overline{Z}\right)$$
$$+ \sigma\left(Y^{-1}\,{}^t(dW)\,d\overline{W}\right)$$
$$\sigma\left(Y^{-1}dZ\,Y^{-1}\,{}^t(d\overline{W})\,V + Y^{-1}d\overline{Z}\,Y^{-1}\,{}^t(dW)\,V\right)$$

is a Riemannian metric on the Siegel-Jacobi space $H_{g,h}$ which is invariant under the action (1.2) of the Jacobi group $G_{g,h}^J$. Also the above metric is a Kähler metric. The Laplace-Beltrami operator $\Delta_{g,h}$ of the Siegel-Jacobi space $(H_{g,h}, ds_{g,h}^2)$ is given by

$$(10.4) \quad \begin{aligned} \Delta_{h,g} = \quad & 4\,\sigma\left(Y\frac{\partial}{\partial Z}Y\frac{\partial}{\partial \overline{Z}}\right) + 4\,\sigma\left(Y\frac{\partial}{\partial W}\,{}^t\left(\frac{\partial}{\partial \overline{W}}\right)\right) \\ & + 4\,\sigma\left(\frac{\partial}{\partial W}V\frac{\partial}{\partial \overline{W}}V\right) \\ & + 4\,\sigma\left(\frac{\partial}{\partial Z}Y\frac{\partial}{\partial \overline{W}}V + \frac{\partial}{\partial \overline{Z}}Y\frac{\partial}{\partial W}V\right). \end{aligned}$$

The following differential form

$$dv := (\det Y)^{-(g+h+1)} [dX] \wedge [dY] \wedge [dU] \wedge [dV]$$

is a $G_{g,h}^J$-invariant volume element on $H_{g,h}$, where

$$[dX] := \wedge_{\mu \leq \nu} dx_{\mu\nu}, \quad [dY] := \wedge_{\mu \leq \nu} dy_{\mu\nu}, \quad [dU] := \wedge_{k,l} du_{kl} \quad and \quad [dV] := \wedge_{k,l} dv_{k,l}.$$

Theorem 10.3. *The automorphism group of $H_{g,h}$ is isomorphic to the group $Sp(g, \mathbb{R}) \ltimes \left(\mathbb{R}^{(h,g)} \times \mathbb{R}^{(h,g)}\right)$ equipped with the multiplication*

$$(M, (\lambda, \mu)) \cdot (M', (\lambda', \mu')) := (MM', (\tilde{\lambda} + \lambda', \tilde{\mu} + \mu')),$$

where $M, M' \in Sp(g, \mathbb{R})$, $\lambda, \mu \in \mathbb{R}^{(h,g)}$ and $(\tilde{\lambda}, \tilde{\mu}) := (\lambda, \mu)M'$.

Theorem 10.4. *The scalar curvature of the Siegel-Jacobi space $(H_1 \times \mathbb{C}, ds^2)$ is -3.*

We note that according to Theorem 2, the metric ds^2 is given by

$$(10.5) \quad ds^2 := ds_{1,1}^2 = \quad \frac{y+v^2}{y^3}\left(dx^2 + dy^2\right) + \frac{1}{y}\left(du^2 + dv^2\right) \\ - \frac{2v}{y^2}\left(dxdu + dydv\right)$$

on $H_1 \times \mathbb{C}$ which is invariant under the action (2.1) of the Jacobi group $G_{1,1}^J = SL(2, \mathbb{R}) \ltimes H_{\mathbb{R}}^{(1,1)}$, where $z = x + iy \in H_1$ and $w = u + iv \in \mathbb{C}$ with x, y, u, v real coordinates.

Remark 10.5. The Poincaré upper half plane H_1 is a two dimensional Riemannian manifold with the Poincaré metric

$$ds_0^2 := \frac{dx^2 + dy^2}{y^2}, \quad z = x + iy \in H_1 \text{ with } x, y \text{ real.}$$

It is easy to see that the Gaussian curvature is -1 everywhere and H_1 is an *Einstein manifold*. In fact, if we denote by $S_0(X, Y)$ the Ricci curvature of (H_1, ds_0^2), then we have

$$S_0(X, Y) = -g_0(X, Y) \quad \text{for all } X, Y \in \mathcal{X}(H_1),$$

where $\mathcal{X}(H_1)$ denotes the algebra of all smooth vector fields on H_1 and $g_0(X, Y)$ is the inner product on the tangent bundle $T(H_1)$ induced by the Poincaré metric ds_0^2. But the Siegel-Jacobi space $H_1 \times \mathbb{C}$ is *not* an Einstein manifold. Indeed, if we denote by $S(X, Y)$ the Ricci curvature of $(H_1 \times \mathbb{C}, ds^2)$ and $E_1 := \frac{\partial}{\partial x}$, we can see without difficulty that there does not exist a constant c such that

$$S(E_1, E_1) \,=\, c\, g(E_1, E_1) \,=\, c\, g_{11} \,=\, c\, \frac{y + v^2}{y^3},.$$

where $g = (g_{ij})$ is the inner product on the tangent bundle $T(H_1 \times \mathbb{C})$ induced by the metric (10.5).

Now we will introduce the notion of *Maass-Jacobi forms*.

Definition 10.6. A smooth function $f \,:\, H_{g,h} \longrightarrow \mathbb{C}$ is called a *Maass-Jacobi form* on $H_{g,h}$ if f satisfies the following conditions (MJ1)-(MJ3) :

(MJ1) f is invariant under $\Gamma_{g,h}^J := \Gamma_g \ltimes H_Z^{(g,h)}$.

(MJ2) f is an eigenfunction of the Lapalce-Beltrami operator $\Delta_{n,m}$.

(MJ3) f has a polynomial growth.

Here $\Gamma_g := Sp(g, Z)$ denotes the Siegel modular group of degree g and and

$$H_Z^{(g,h)} := \left\{ (\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(g,h)} \mid \lambda, \mu, \kappa \text{ integral} \right\}.$$

For more details on Maass-Jacobi forms in the case $g = h = 1$, we refer to [89].

11. Final Remarks

In [32] and [34], Gritsenko, Hulek and Sankaran gave applications of Jacobi forms of degree 1 in the study of the moduli space of abelian surfaces with a certain polarization. We refer to [7],[9],[11],[61],[62] for the representation theory of the Jacobi group.

Appendix A. Subvarieties of the Siegel Modular Variety

Here we assume that the ground field is the complex number field $\mathbb{C}$.

Definition A.1. A nonsingular variety X is said to be *rational* if X is birational to a projective space $P^n(\mathbb{C})$ for some integer n. A nonsingular variety X is said to be *stably rational* if $X \times P^k(\mathbb{C})$ is birational to $P^N(\mathbb{C})$ for certain nonnegative integers k and N. A nonsingular variety X is called *unirational* if there exists a dominant rational map $\varphi \,:\, P^n(\mathbb{C}) \longrightarrow X$ for a certain positive integer n, equivalently if the function field $\mathbb{C}(X)$ of X can be embedded in a purely transcendental extension $\mathbb{C}(z_1, \cdots, z_n)$ of $\mathbb{C}$.

Remarks A.2.   (1) It is easy to see that the rationality implies the stably rationality and that the stably rationality implies the unirationality.

(2) If X is a Riemann surface or a complex surface, then the notions of rationality, stably rationality and unirationality are equivalent one another.

(3) Griffiths and Clemens(cf. Ann. of Math. 95(1972), 281-356) showed that most of cubic threefolds in $P^4(\mathbb{C})$ are unirational but *not* rational.

The following natural questions arise :

QUESTION 1.   Is a stably rational variety *rational*? Indeed, the question was raised by Bogomolov.

QUESTION 2.   Is a general hypersurface $X \subset P^{n+1}(\mathbb{C})$ of degree $d \leq n+1$ *unirational*?

Definition A.3.   Let X be a nonsingular variety of dimension n and let K_X be the canonical divisor of X. For each positive integer $m \in \mathbb{Z}^+$, we define the *m-genus* $P_m(X)$ of X by

$$P_m(X) := \dim_{\mathbb{C}} H^0(X, \mathcal{O}(mK_X)).$$

The number $p_g(X) := P_1(X)$ is called the *geometric genus* of X. We let

$$N(X) := \left\{ m \in \mathbb{Z}^+ \,|\, P_m(X) \geq 1 \right\}.$$

For the present, we assume that $N(X)$ is nonempty. For each $m \in N(X)$, we let $\{\phi_0, \cdots, \phi_{N_m}\}$ be a basis of the vector space $H^0(X, \mathcal{O}(mK_X))$. Then we have the mapping $\Phi_{mK_X} : X \longrightarrow P^{N_m}(\mathbb{C})$ by

$$\Phi_{mK_X}(z) := (\phi_0(z) : \cdots : \phi_{N^m}(z)), \quad z \in X.$$

We define the *Kodaira dimension* $\kappa(X)$ of X by

$$\kappa(X) := \max \left\{ \dim_{\mathbb{C}} \Phi_{mK_X}(X) \,|\, m \in N(X) \right\}.$$

If $N(X)$ is empty, we put $\kappa(X) := -\infty$. Obviously $\kappa(X) \leq \dim_{\mathbb{C}} X$. A nonsingular variety X is said to be *of general type* if $\kappa(X) = \dim_{\mathbb{C}} X$. A singular variety Y in general is said to be rational, stably rational, unirational or of general type if any nonsingular model X of Y is rational, stably rational, unirational or of general type respectively. We define

$$P_m(Y) := P_m(X) \quad \text{and} \quad \kappa(Y) := \kappa(X).$$

A variety Y of dimension n is said to be *of logarithmic general type* if there exists a smooth compactification $\tilde{Y}$ of Y such that $D := \tilde{Y} - Y$ is a divisor with normal crossings only and the transcendence degree of the logarithmic canonical ring

$$\oplus_{m=0}^{\infty} H^0(\tilde{Y}, m(K_{\tilde{Y}} + [D]))$$

is $n + 1$, i.e., the *logarithmic Kodaira dimension* of Y is n. We observe that the notion of being of logarithmic general type is weaker than that of being of general type.

Let $A_g := \Gamma_g \backslash H_g$ be the Siegel modular variety of degree g, that is, the moduli space of principally polarized abelian varieties of dimension g. So far it has been proved that A_g is of general type for $g \geq 7$. At first Freitag [24] proved this fact when g is a multiple of 24. Tai [60] proved this for $g \geq 9$ and Mumford [46] proved this fact for $g \geq 7$. On the other hand, A_g is known to be unirational for $g \leq 5$: Donagi [16] for $g = 5$, Clemens [15] for $g = 4$ and classical for $g \leq 3$. For $g = 3$, using the moduli theory of curves, Riemann [53], Weber [65] and Frobenius [28] showed that $A_3(2) := \Gamma_3(2) \backslash H_3$ is a rational variety and moreover gave 6 generators of the modular function field $K(\Gamma_3(2))$ written explicitly in terms of derivatives of odd theta functions at the origin. So A_3 is a unirational variety with a Galois covering of a rational variety of degree $[\Gamma_3 : \Gamma_3(2)] = 1,451,520$. Here $\Gamma_3(2)$ denotes the principal congruence subgroup of Γ_3 of level 2. Furthermore it was shown that A_3 is stably rational(cf. [38], [12]). For a positive integer k, we let $\Gamma_g(k)$ be the principal congruence subgroup of Γ_g of level k. Let $A_g(k)$ be the moduli space of abelian varieties of dimension g with k-level structure. It is classically known that $A_g(k)$ is of logarithmic general type for $k \geq 3$(cf. [45]). Wang [64] proved that $A_2(k)$ is of general type for $k \geq 4$. On the other hand, van der Geer [29] showed that $A_2(3)$ is rational. The remaining unsolved problems are summarized as follows :

Problem 1.   Is A_3 rational ?

Problem 2.   Are A_4, A_5 stably rational or rational ?

Problem 3.   Discuss the (uni)rationality of A_6.

Problem 4.   What type of varieties are $A_g(k)$ for $g \geq 3$ and $k \geq 2$?

We already mentioned that A_g is of general type if $g \geq 7$. It is natural to ask if the subvarieties of A_g $(g \geq 7)$ are of general type, in particular the subvarieties of A_g of codimension one. Freitag [Fr3] showed that there exists a certain bound g_0 such that for $g \geq g_0$, each irreducible subvariety of A_g of codimension one is of general type. Weissauer [Wei2] proved that every irreducible divisor of A_g is of general type for $g \geq 10$. Moreover he proved that every subvariety of codimension $\leq g - 13$ in A_g is of general type for $g \geq 13$. We observe that the smallest known codimension for which there exist subvarieties of A_g for large g which are not of general type is $g - 1$. $A_1 \times A_{g-1}$ is a subvariety of A_g of codimension $g - 1$ which is not of general type.

Remark A.4.   Let $\mathcal{M}_g$ be the coarse moduli space of curves of genus g over $\mathbb{C}$. Then $\mathcal{M}_g$ is an analytic subvariety of A_g of dimension $3g - 3$. It is known that $\mathcal{M}_g$ is unirational for $g \leq 10$. So the Kodaira dimension $\kappa(\mathcal{M}_g)$ of $\mathcal{M}_g$ is $-\infty$ for $g \leq 10$. Harris and Mumford [H-M] proved that $\mathcal{M}_g$ is of general type for odd g

with $g \geq 25$ and $\kappa(\mathcal{M}_{23}) \geq 0$.

Appendix B. Singular Modular Forms

Let ρ be a rational representation of $GL(g, \mathbb{C})$ on a finite dimensional complex vector space V_ρ. A holomorphic function $f : H_g \longrightarrow V_\rho$ with values in V_ρ is called a modular form of type ρ if it satisfies

$$f(M < Z >) = \rho(CZ + D)f(Z)$$

for all $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$ and $Z \in H_g$. We denote by $[\Gamma_g, \rho]$ the vector space of all modular forms of type ρ. A modular form $f \in [\Gamma_g, \rho]$ of type ρ has a Fourier series

$$f(Z) = \sum_{T \geq 0} a(T)e^{2\pi i(TZ)}, \quad Z \in H_g,$$

where T runs over the set of all semipositive half-integral symmetric matrices of degree g. A modular form f of type ρ is said to be *singular* if a Fourier coefficient $a(T)$ vanishes unless $\det(T) = 0$.

Freitag [25] proved that every singular modular form can be written as a finite linear combination of theta series with harmonic coefficients and proposed the problem to characterize singular modular forms. Weissauer [66] gave the following criterion.

Theorem B.1. *Let ρ be an irreducible rational representation of $GL(g, \mathbb{C})$ with its highest weight $(\lambda_1, \cdots, \lambda_g)$. Let f be a modular form of type ρ. Then the following are equivalent:*

(a) f is singular.

(b) $2\lambda_g < g$.

Now we describe how the concept of singular modular forms is closely related to the geometry of the Siegel modular variety. Let X be the Satake compactification of the Siegel modular variety $A_g = \Gamma_g \backslash H_g$. Then A_g is embedded in X as a quasiprojective algebraic subvariety of codimension g. Let X_s be the smooth part of A_g and $\tilde{X}$ the desingularization of X. Without loss of generality, we assume $X_s \subset \tilde{X}$. Let $\Omega^p(\tilde{X})$ (resp. $\Omega^p(X_s)$) be the space of holomorphic p-form on $\tilde{X}$ (resp. X_s). Freitag and Pommerening [27] showed that if $g > 1$, then the restriction map

$$\Omega^p(\tilde{X}) \longrightarrow \Omega^p(X_s)$$

is an isomorphism for $p < \dim_{\mathbb{C}} \tilde{X} = \frac{g(g+1)}{2}$. Since the singular part of A_g is at least codimension 2 for $g > 1$, we have an isomorphism

$$\Omega^p(\tilde{X}) \cong \Omega^p(H_g)^{\Gamma_g}.$$

Here $\Omega^p(H_g)^{\Gamma_g}$ denotes the space of Γ_g-invariant holomorphic p-forms on H_g. Let $\mathrm{Sym}^2(\mathbb{C}^g)$ be the symmetric power of the canonical representation of $GL(g,\mathbb{C})$ on $\mathbb{C}^n$. Then we have an isomorphism

$$\Omega^p(H_g)^{\Gamma_g} \longrightarrow [\Gamma_g, \wedge^p\mathrm{Sym}^2(\mathbb{C}^g)].$$

Theorem B.2([66]). *Let ρ_α be the irreducible representation of $GL(g,\mathbb{C})$ with highest weight*

$$(g+1, \cdots, g+1, g-\alpha, \cdots, g-\alpha)$$

such that $\mathrm{corank}(\rho_\alpha) = \alpha$ *for* $1 \le \alpha \le g$. *If* $\alpha = -1$, *we let* $\rho_\alpha = (g+1, \cdots, g+1)$. *Then*

$$\Omega^p(H_g)^{\Gamma_g} = \begin{cases} [\Gamma_g, \rho_\alpha], & \text{if } p = \frac{g(g+1)}{2} - \frac{\alpha(\alpha+1)}{2} \\ 0, & \text{otherwise.} \end{cases}$$

Remark B.3. *If* $2\alpha > g$, *then any* $f \in [\Gamma_g, \rho_\alpha]$ *is singular. Thus if* $p < \frac{g(3g+2)}{8}$, *then any* Γ_g*-invariant holomorphic p-form on H_g can be expressed in terms of vector valued theta series with harmonic coefficients. It can be shown with a suitable modification that the just mentioned statement holds for a sufficiently small congruence subgroup of* Γ_g.

Thus the natural question is to ask how to determine the Γ_g-invariant holomorphic p-forms on H_g for the nonsingular range $\dfrac{g(3g+2)}{8} \le p \le \dfrac{g(g+1)}{2}$. Weissauer [68] answered the above question for $g = 2$. For $g > 2$, the above question is still open. It is well know that the vector space of vector valued modular forms of type ρ is finite dimensional. The computation or the estimate of the dimension of $\Omega^p(H_g)^{\Gamma_g}$ is interesting because its dimension is finite even though the quotient space A_g is noncompact.

Finally we will mention the results due to Weisauer [67]. We let Γ be a congruence subgroup of Γ_2. According to Theorem B.2, Γ-invariant holomorphic forms in $\Omega^2(H_2)^\Gamma$ are corresponded to modular forms of type $(3,1)$. We note that these invariant holomorphic 2-forms are contained in the *nonsingular range*. And if these modular forms are not cusp forms, they are mapped under the Siegel Φ-operator to cusp forms of weight 3 with respect to some congruence subgroup (dependent on Γ) of the elliptic modular group. Since there are finitely many cusps, it is easy to deal with these modular forms in the adelic version. Observing these facts, he showed that any 2-holomorphic form on $\Gamma\backslash H_2$ can be expressed in terms of theta series with harmonic coefficients associated to binary positive definite quadratic forms. Moreover he showed that $H^2(\Gamma\backslash H_2, \mathbb{C})$ has a pure Hodge structure and that the Tate conjecture holds for a suitable compactification of $\Gamma\backslash H_2$. If $g \ge 3$, for a congruence subgroup Γ of Γ_g it is difficult to compute the cohomology groups $H^*(\Gamma\backslash H_g, \mathbb{C})$ because $\Gamma\backslash H_g$ is noncompact and highly singular. Therefore in order to study their structure, it is natural to ask if they have pure Hodge structures or mixed Hodge structures.

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