Geometry and Arithmetic on Siegel-Jacobi Space

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1. The Jacobi Group

For a given fixed positive integer n, we let

$$\mathbb{H}_n = \{\Omega \in \mathbb{C}^{(n,n)} \mid \Omega = {}^t\Omega,\ \operatorname{Im}\Omega > 0\}$$

be the Siegel upper half plane of degree n and

$$\operatorname{Sp}(n,\mathbb{R}) = \{M \in \mathbb{R}^{(2n,2n)} \mid {}^tMJ_nM = J_n\}$$

be the symplectic group of degree n, where

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

We see that $\operatorname{Sp}(n,\mathbb{R})$ acts on $\mathbb{H}_n$ transitively by

$$M \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1},$$

where $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \operatorname{Sp}(n,\mathbb{R})$ and $\Omega \in \mathbb{H}_n$.

- **Jacobi group and Siegel-Jacobi space**

$$G^J := \mathrm{Sp}(n,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)}$$

$$\mathbb{H}_{n,m} := \mathbb{H}_n \times \mathbb{C}^{(m,n)}$$

$$(M,(\lambda,\mu;\kappa)) \cdot (\Omega, Z) = (M\cdot\Omega, (Z+\lambda\Omega+\mu)(C\Omega+D)^{-1})$$

$$\mathbb{H}_{n,m} = G^J/K^J, \qquad K^J = U(n)\times S(n,\mathbb{R})$$

The Siegel-Jacobi space is not a reductive symmetric space

$S(n,\mathbb{R})$ is the space of all $n\times n$ symmetric real matrices

2. Invariant Metrics on Siegel-Jacobi Space

- Siegel Metric (1943, AJM)

$$ds_n^2 = \sigma\left(Y^{-1} d\Omega\, Y^{-1} d\overline{\Omega}\right)$$

- Laplacian of Siegel Metric
  Hans Maass (1953, Math. Ann.)

$$\Delta_n = 4\sigma\left(Y\ {}^t\!\left(Y \frac{\partial}{\partial \overline{\Omega}}\right) \frac{\partial}{\partial \Omega}\right)$$

- Invariant Metric (J.-H. Yang: 2007, JNT)

$$ds^2_{n,m;A,B} = A\sigma(Y^{-1}d\Omega Y^{-1}d\overline{\Omega}) + B\{\sigma(Y^{-1}\,{}^tVVY^{-1}d\Omega Y^{-1}d\overline{\Omega}) + \sigma(Y^{-1}\,{}^t(dZ)d\overline{Z}) - \sigma(VY^{-1}d\Omega Y^{-1}\,{}^t(d\overline{Z})) - \sigma(VY^{-1}d\overline{\Omega}Y^{-1}\,{}^t(dZ))\}$$

- Laplacian (J.-H. Yang: 2007, JNT)

$$\Delta_{n,m;A,B} = \frac{4}{A}\left\{\sigma\left(Y\,{}^t\left(Y\frac{\partial}{\partial\overline{\Omega}}\right)\frac{\partial}{\partial\Omega}\right) + \sigma\left(VY^{-1}\,{}^tV\,{}^t\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial Z}\right) + \sigma\left(V\,{}^t\left(Y\frac{\partial}{\partial\overline{\Omega}}\right)\frac{\partial}{\partial Z}\right) + \sigma\left({}^tV\,{}^t\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial\Omega}\right)\right\} + \frac{4}{B}\sigma\left(Y\frac{\partial}{\partial Z}\,{}^t\left(\frac{\partial}{\partial\overline{Z}}\right)\right).$$

Problem 1. (a) Find an explicit **geodesic**

(2) Compute the **distance** between two points explicitly

(3) Compute the **sectional curvature**

Problem 2. Develop the **spectral theory** of the **Laplacian** for an **arithmetic** subgroup of the **Siegel-Jacobi modular group**

$$\Gamma_{n,m} := Sp(n,\mathbb{Z}) \ltimes H_{\mathbb{Z}}^{(n,m)}$$

Remark: In the case that **n=m=A=B=1**, **Erik Balslev** developed the spectral theory for $\Gamma_{1,1}$ and $\Gamma_1(2) \ltimes H_{\mathbb{Z}}^{(1,1)}$. And he also showed that the set of all eigenvalues of the Laplacian satisfies the **Weyl law** for the above special arithmetic subgroups [2012].

3. Invariant Differential Operators

Let $\mathbb{D}(\mathbb{H}_{n,m})$ be the algebra of all invariant differential operators on the S-J space.

Problem 1. Find a list of generators of $\mathbb{D}(\mathbb{H}_{n,m})$
Is $\mathbb{D}(\mathbb{H}_{n,m})$ finitely generators ?

Problem 2. Find all the relations among a given list of generators of $\mathbb{D}(\mathbb{H}_{n,m})$

Problem 3. Find the center of $\mathbb{D}(\mathbb{H}_{n,m})$

$U(n)$ acts on $T_{n,m} := S(n,\mathbb{C}) \times \mathbb{C}^{(m,n)}$

$$k \cdot (\omega, z) = (k\,\omega\,{}^t k,\, z\,{}^t k), \quad k \in U(n),\ (\omega, z) \in T_{n,m}$$

Then it induces naturally the action of on the polynomial algebra $\mathrm{Pol}(T_{n,m})$

Let $\mathrm{Pol}(T_{n,m})^{U(n)}$ be the K-invariants of $\mathrm{Pol}(T_{n,m})$

Then we have the natural canonical linear bijection

$$\Theta_{n,m} : \mathrm{Pol}(T_{n,m})^{U(n)} \longrightarrow \mathbb{D}(\mathbb{H}_{n,m})$$

Problem 4. Find a list of generators of $\mathrm{Pol}(T_{n,m})^{U(n)}$

Is $\mathrm{Pol}(T_{n,m})^{U(n)}$ finitely generated ?

Problem 5. Find all the relations among a given list of generators of $\mathrm{Pol}(T_{n,m})^{U(n)}$

Problem 6. Are the images of a given list of generators $\mathrm{Pol}(T_{n,m})^{U(n)}$ under $\Theta_{n,m}$ a list of generators of $\mathbb{D}(\mathbb{H}_{n,m})$?

Remarks:

(1) Problem 4 and Problem 5 were solved :

M. Itoh, H. Ochiai and J.-H. Yang
Invariant Differential Operators on Siegel-Jacobi Space [2013]

(2) In the case that $n=m=1$, all the above problems were completely solved by Itoh, Ochiai and Yang in the same paper

4. The Partial Cayley Transform

Let

$$G_* = \mathrm{SU}(n,n) \cap \mathrm{Sp}(n,\mathbb{C})$$

be the symplectic group and

$$\mathbb{D}_n = \{W \in \mathbb{C}^{(n,n)} \mid W = {}^tW,\ I_n - \overline{W}W > 0\}$$

be the generalized unit disk. Then G_* acts on $\mathbb{D}_n$ transitively by

$$\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \cdot W = (PW + Q)(\overline{Q}W + \overline{P})^{-1},$$

where $\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \in G_*$ and $W \in \mathbb{D}_n$. Using the Cayley transform of $\mathbb{D}_n$ onto $\mathbb{H}_n$, we can see that

$$ds_*^2 = 4\sigma((I_n - W\overline{W})^{-1}dW(I_n - \overline{W}W)^{-1}d\overline{W}) \tag{1.7}$$

is a G_*-invariant Kähler metric on $\mathbb{D}_n$ (see [6]) and Maass [4] showed that its Laplacian is given by

$$\Delta_* = \sigma\left((I_n - W\overline{W})\ {}^t\left((I_n - W\overline{W})\frac{\partial}{\partial \overline{W}}\right)\frac{\partial}{\partial W}\right). \tag{1.8}$$

Let

$$G_*^J = \left\{ \left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, (\xi, \overline{\xi}; i\kappa) \right) \,\middle|\, \begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \in G_*,\ \xi \in \mathbb{C}^{(m,n)},\ \kappa \in \mathbb{R}^{(m,m)} \right\}$$

be the Jacobi group with the following multiplication:

$$\left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, (\xi, \overline{\xi}; i\kappa) \right) \cdot \left(\begin{pmatrix} P' & Q' \\ \overline{Q'} & \overline{P'} \end{pmatrix}, (\xi', \overline{\xi'}; i\kappa') \right)$$
$$= \left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \begin{pmatrix} P' & Q' \\ \overline{Q'} & \overline{P'} \end{pmatrix}, (\widetilde{\xi} + \xi', \widetilde{\overline{\xi'}} + \overline{\xi'}; i\kappa + i\kappa' + \widetilde{\xi}\,{}^t\overline{\xi'} - \widetilde{\overline{\xi'}}\,{}^t\xi') \right),$$

where $\widetilde{\xi} = \xi P' + \overline{\xi}\,\overline{Q'}$ and $\widetilde{\overline{\xi}} = \xi Q' + \overline{\xi}\,\overline{P'}$. Then we have the natural action of G_*^J on the Siegel-Jacobi disk $\mathbb{D}_n \times \mathbb{C}^{(m,n)}$ (see (2.6)) given by

$$\left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, (\xi, \overline{\xi}; i\kappa) \right) \cdot (W, \eta) = \left((PW + Q)(\overline{Q}W + \overline{P})^{-1}, (\eta + \xi W + \overline{\xi})(\overline{Q}W + \overline{P})^{-1} \right), \quad (1.9)$$

where $W \in \mathbb{D}_n$ and $\eta \in \mathbb{C}^{(m,n)}$.

Theorem (J.-H. Yang: 2008, JKMS)

The action of G^J on $\mathbb{H}_{n,m}$ is compatible with the action of G^J_* on $\mathbb{D}_{n,m}$ through the **partial Cayley transform** defined by

$$\Phi(W,\eta) := \left(\mathrm{i}(I_n+W)(I_n-W)^{-1},\, 2\mathrm{i}\eta(I_n-W)^{-1}\right).$$

The inverse of Φ is given by

$$\Phi^{-1}(\Omega,Z) = \left((\Omega-\mathrm{i}I_n)(\Omega+\mathrm{i}I_n)^{-1},\, Z(\Omega+\mathrm{i}I_n)^{-1}\right).$$

5. Invariant Metrics on Siegel-Jacobi Disk

[1] Siegel Metric

$$ds^2_{\mathbb{D}_n;A} = 4\,A\,\mathrm{tr}\Big((I_n - W\overline{W})^{-1} dW (I_n - \overline{W}W)^{-1} d\overline{W} \Big)$$

[2] Laplacian of Siegel Metric (H. Maass)

$$\Delta_{\mathbb{D}_n;A} = \frac{1}{A}\,\mathrm{tr}\left((I_n - W\overline{W})\, {}^t\!\left((I_n - W\overline{W}) \frac{\partial}{\partial \overline{W}} \right) \frac{\partial}{\partial W} \right)$$

Theorem 1.3 *For any two positive real numbers* A *and* B, *the following metric* $d\tilde{s}^2_{n,m;A,B}$ *defined by*

$$
\begin{aligned}
d\tilde{s}^2_{n,m;A,B} = {} & 4A\sigma((I_n - W\overline{W})^{-1}dW(I_n - \overline{W}W)^{-1}d\overline{W}) + 4B\{\sigma((I_n - W\overline{W})^{-1}\,{}^t(d\eta)d\overline{\eta})\\
& + \sigma((\eta\overline{W} - \overline{\eta})(I_n - W\overline{W})^{-1}dW(I_n - \overline{W}W)^{-1}\,{}^t(d\overline{\eta}))\\
& + \sigma((\overline{\eta}W - \eta)(I_n - \overline{W}W)^{-1}d\overline{W}(I_n - W\overline{W})^{-1}\,{}^t(d\eta))\\
& - \sigma((I_n - W\overline{W})^{-1}\,{}^t\eta\eta(I_n - \overline{W}W)^{-1}\overline{W}dW(I_n - \overline{W}W)^{-1}d\overline{W})\\
& - \sigma(W(I_n - \overline{W}W)^{-1}\,{}^t\overline{\eta}\,\overline{\eta}(I_n - W\overline{W})^{-1}dW(I_n - \overline{W}W)^{-1}d\overline{W})\\
& + \sigma((I_n - W\overline{W})^{-1}\,{}^t\eta\overline{\eta}(I_n - W\overline{W})^{-1}dW(I_n - \overline{W}W)^{-1}d\overline{W})\\
& + \sigma((I_n - \overline{W})^{-1}\,{}^t\overline{\eta}\eta\overline{W}(I_n - W\overline{W})^{-1}dW(I_n - \overline{W}W)^{-1}d\overline{W})\\
& + \sigma((I_n - \overline{W})^{-1}(I_n - W)(I_n - \overline{W}W)^{-1}\,{}^t\overline{\eta}\eta(I_n - \overline{W}W)^{-1}(I_n - \overline{W})(I_n - W)^{-1}\\
& \times dW(I_n - \overline{W}W)^{-1}d\overline{W}) - \sigma((I_n - W\overline{W})^{-1}(I_n - W)(I_n - \overline{W})^{-1}\,{}^t\overline{\eta}\eta\\
& \times (I_n - W)^{-1}dW(I_n - \overline{W}W)^{-1}d\overline{W})\}
\end{aligned}
$$

is a Riemannian metric on $\mathbb{D}_{n,m}$ *which is invariant under the action* (1.9) *of the Jacobi group*

Theorem 1.4 *For any two positive real numbers* A *and* B, *the Laplacian* $\tilde{\Delta}_{n,m;A,B}$ *of* $(\mathbb{D}_{n,m}, d\tilde{s}^2_{n,m;A,B})$ *is given by*

$$\begin{aligned}
\tilde{\Delta}_{n,m;A,B} = \frac{1}{A}\Big\{ & \sigma\left((I_n - W\overline{W})\, {}^t\!\left((I_n - W\overline{W})\frac{\partial}{\partial \overline{W}}\right)\frac{\partial}{\partial W}\right) \\
& + \sigma\left({}^t(\eta - \overline{\eta} W)\, {}^t\!\left(\frac{\partial}{\partial \overline{\eta}}\right)(I_n - \overline{W}W)\frac{\partial}{\partial W}\right) + \sigma\left((\overline{\eta} - \eta\overline{W})\, {}^t\!\left((I_n - W\overline{W})\frac{\partial}{\partial \overline{W}}\right)\frac{\partial}{\partial \eta}\right) \\
& - \sigma\left(\eta\overline{W}(I_n - W\overline{W})^{-1}\, {}^t\eta\, {}^t\!\left(\frac{\partial}{\partial \overline{\eta}}\right)(I_n - \overline{W}W)\frac{\partial}{\partial \eta}\right) \\
& - \sigma\left(\overline{\eta}W(I_n - \overline{W}W)^{-1}\, {}^t\overline{\eta}\, {}^t\!\left(\frac{\partial}{\partial \overline{\eta}}\right)(I_n - \overline{W}W)\frac{\partial}{\partial \eta}\right) \\
& + \sigma\left(\overline{\eta}(I_n - W\overline{W})^{-1}\, {}^t\eta\, {}^t\!\left(\frac{\partial}{\partial \overline{\eta}}\right)(I_n - \overline{W}W)\frac{\partial}{\partial \eta}\right) \\
& + \sigma\left(\eta\overline{W}W(I_n - \overline{W}W)^{-1}\, {}^t\overline{\eta}\, {}^t\!\left(\frac{\partial}{\partial \overline{\eta}}\right)(I_n - \overline{W}W)\frac{\partial}{\partial \eta}\right)\Big\} \\
& + \frac{1}{B}\sigma\left((I_n - \overline{W}W)\frac{\partial}{\partial \eta}\, {}^t\!\left(\frac{\partial}{\partial \overline{\eta}}\right)\right).
\end{aligned}$$

Problem 1. Develop the theory of harmonic analysis on Siegel-Jacobi disk. [Remark : The theory of harmonic analysis on the unit disk of complex dimension **one** explicitly around the 1970s by **S. Helgason**]

Problem 2. Develop the theory of harmonic analysis on the quotient of Siegel-Jacobi disk by an arithmetic subgroup of the Siegel-Jacobi modular group.

6. A Fundamental Domain for Siegel-Jacobi Space

Let E_{kj} be the $m \times n$ matrix with entry 1 where the k-th row and the j-th column meet, and all other entries 0.
For an element $\Omega \in \mathbb{H}_n$, we set

$$F_{kj}(\Omega) := E_{kj}\Omega, \quad 1 \leq k \leq m,\ 1 \leq j \leq n.$$

Let $\mathcal{F}_n$ be Siegel's fundamental domain for $\Gamma_n \backslash \mathbb{H}_n$.
For each $\Omega \in \mathcal{F}_n$, we define the subset P_Ω of $\mathbb{C}^{(m,n)}$ by

$$P_\Omega = \left\{ \sum_{k=1}^{m} \sum_{j=1}^{n} \lambda_{kj} E_{kj} + \sum_{k=1}^{m} \sum_{j=1}^{n} \mu_{kj} F_{kj}(\Omega) \,\middle|\, 0 \leq \lambda_{kj}, \mu_{kj} \leq 1 \right\}.$$

For each $\Omega \in \mathcal{F}_n$, we define the subset D_Ω of $\mathbb{H}_{n,m}$ by

$$D_\Omega = \{(\Omega, Z) \in \mathbb{H}_{n,m} \mid Z \in P_\Omega\}.$$

Theorem (J.-H. Yang: 2006). The set

$$\mathcal{F}_{n,m} := \bigcup_{\Omega \in \mathcal{F}_n} D_\Omega$$

is a fundamental domain for $\Gamma_{n,m} \backslash \mathbb{H}_{n,m}$.

7. The Canonical Automorphic factor for Jacobi Group

Let ρ be a rational representation of $GL(n,\mathbb{C})$ on a finite dimensional complex vector space V_ρ. Let $\mathcal{M} \in \mathbb{R}^{(m,m)}$ be a symmetric half-integral semi-positive definite matrix of degree m.

The canonical automorphic factor

$$J_{\rho,\mathcal{M}} : G^J \times \mathbb{H}_{n,m} \longrightarrow GL(V_\rho)$$

for G^J on $\mathbb{H}_{n,m}$ is given as follows:

If $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R}),\ (\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)}$ and $(\Omega, Z) \in \mathbb{H}_{n,m}$,

$$
\begin{aligned}
& J_{\rho,\mathcal{M}}((g,(\lambda,\mu;\kappa)),(\Omega,Z)) \\
= \;& e^{2\pi i\,\sigma\left(\mathcal{M}(Z+\lambda\Omega+\mu)(C\Omega+D)^{-1}C\,{}^t(Z+\lambda\Omega+\mu)\right)} \\
& \times e^{-2\pi i\,\sigma\left(\mathcal{M}(\lambda\Omega\,{}^t\lambda+2\lambda\,{}^tZ+\kappa+\mu\,{}^t\lambda)\right)}\rho(C\,\Omega+D)
\end{aligned}
$$

Let $C^\infty(\mathbb{H}_{n,m},V_\rho)$ be the algebra of all C^∞ functions on $\mathbb{H}_{n,m}$ with values in V_ρ. For $f\in C^\infty(\mathbb{H}_{n,m},V_\rho)$, we define

$$
\begin{aligned}
& \left(f\,|_{\rho,\mathcal{M}}[(g,(\lambda,\mu;\kappa))]\right)(\Omega,Z) \\
= \;& J_{\rho,\mathcal{M}}((g,(\lambda,\mu;\kappa)),(\Omega,Z))^{-1} \\
& f\left(g\cdot\Omega,(Z+\lambda\,\Omega+\mu)(C\,\Omega+D)^{-1}\right).
\end{aligned}
$$

Definition. Let ρ and $\mathcal{M}$ be as above. Let

$$H_{\mathbb{Z}}^{(n,m)} := \left\{ (\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(n,m)} \mid \lambda, \mu, \kappa \text{ integral } \right\}$$

be the discrete subgroup of $H_{\mathbb{R}}^{(n,m)}$. A Jacobi form of index $\mathcal{M}$ with respect to ρ on a subgroup Γ of Γ_n of finite index is a holomorphic function $f \in C^\infty(\mathbb{H}_{n,m}, V_\rho)$ satisfying the following conditions (A) and (B):

(A) $f|_{\rho,\mathcal{M}}[\tilde{\gamma}] = f$ for all $\tilde{\gamma} \in \widetilde{\Gamma} := \Gamma \ltimes H_{\mathbb{Z}}^{(n,m)}$.

(B) For each $M \in \Gamma_n$, $f|_{\rho,\mathcal{M}}[M]$ has a Fourier expansion of the following form :

$$(f|_{\rho,\mathcal{M}}[M])(\Omega, Z) = \sum_{\substack{T = {}^tT \geq 0 \\ \text{half-integral}}} \sum_{R \in \mathbb{Z}^{(n,m)}} c(T,R) \cdot e^{\frac{2\pi i}{\lambda_\Gamma} \sigma(T\Omega)} \cdot e^{2\pi i\, \sigma(RZ)}$$

with $\lambda_\Gamma (\neq 0) \in \mathbb{Z}$ and $c(T,R) \neq 0$ only if $\begin{pmatrix} \frac{1}{\lambda_\Gamma} T & \frac{1}{2} R \\ \frac{1}{2} {}^tR & \mathcal{M} \end{pmatrix} \geq 0$.

8. Singular Jacobi Forms

Definition. A Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma_n)$ is said to be cuspidal if $\begin{pmatrix} T & \frac{1}{2}R \\ \frac{1}{2}\,{}^tR & \mathcal{M} \end{pmatrix} > 0$ for any T, R with $c(T,R) \neq 0$.

Definition. A Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma_n)$ is said to be singular if it admits a Fourier expansion such that a Fourier coefficient $c(T,R)$ vanishes unless $\det \begin{pmatrix} T & \frac{1}{2}R \\ \frac{1}{2}\,{}^tR & \mathcal{M} \end{pmatrix} = 0$.

Let $\mathscr{P}_{n,m} = \mathscr{P}_n \times \mathbb{R}^{(m,n)}$ be the Minkowski-Euclid space, where $\mathscr{P}_n$ is the open cone consisting of positive symmetric $n \times n$ real matrices.

For a variable $(Y,V) \in \mathscr{P}_{n,m}$ with $Y \in \mathscr{P}_n$ and $V \in \mathbb{R}^{(m,n)}$, we put

$$Y = (y_{\mu\nu}) \text{ with } y_{\mu\nu} = y_{\nu\mu}, \quad V = (v_{kl}),$$
$$\frac{\partial}{\partial Y} = \left(\frac{1+\delta_{\mu\nu}}{2} \frac{\partial}{\partial y_{\mu\nu}} \right), \quad \frac{\partial}{\partial V} = \left(\frac{\partial}{\partial v_{kl}} \right),$$

where $1 \leq \mu, \nu, l \leq n$ and $1 \leq k \leq m$.

We define the following differential operator

$$M_{n,m,\mathcal{M}} := \det(Y) \cdot \det \left(\frac{\partial}{\partial Y} + \frac{1}{8\pi} {}^t\!\left(\frac{\partial}{\partial V} \right) \mathcal{M}^{-1} \frac{\partial}{\partial V} \right)$$

Theorem (J.-H. Yang: 1995) Let $f \in J_{\rho,\mathcal{M}}(\Gamma_n)$. Then

f is singular if and only if $M_{n,m,\mathcal{M}} f = 0$.

Theorem (J.-H. Yang: 1995) Let $f(\neq 0) \in J_{\rho,\mathcal{M}}(\Gamma_n)$.
Then

f is singular if and only if $2\,k(\rho) < n + m$.

9. The Siegel-Jacobi Operator

Let ρ be a rational representation of $GL(n,\mathbb{C})$ on a finite dimensional vector space V_ρ. For a positive integer $r < n$, we let $\rho^{(r)} : GL(r,\mathbb{C}) \longrightarrow GL(V_\rho)$ be a rational representation of $GL(r,\mathbb{C})$ defined by

$$\rho^{(r)}(a)v := \rho\left(\begin{pmatrix} a & 0 \\ 0 & iI_{n-r} \end{pmatrix}\right) v, \quad a \in GL(r,\mathbb{C}),\ v \in V_\rho.$$

The **Siegel-Jacobi operator** $\Psi_{n,r} : J_{\rho,\mathcal{M}}(\Gamma_n) \longrightarrow J_{\rho^{(r)},\mathcal{M}}(\Gamma_n)$ is defined by

$$\left(\Psi_{n,r} f\right)(\Omega, Z) := \lim_{t\longrightarrow\infty} f\left(\begin{pmatrix} \Omega & 0 \\ 0 & itI_{n-r} \end{pmatrix}, (Z, 0)\right),$$

where $f \in J_{\rho,\mathcal{M}}(\Gamma_n)$, $\Omega \in \mathbb{H}_r$ and $Z \in \mathbb{C}^{(m,r)}$.

Theorem (J.-H. Yang: 1993)

The action of the Siegel-Jacobi operator on Jacobi forms is compatible with that of the Hecke operator.

Remarks : (a) We can define the concept of stable Jacobi forms using the Siegel-Jacobi operator. The stability of Jacobi forms yields the study of representations of an infinite dimensional symplectic group, an inf. dim. unitary group, an inf. dim. Jacobi group and that of an inf. dim. universal abelian variety.

(b) We may investigate the injectivity and the surjectivity of the Siegel-Jacobi operator.

10. Construction of Modular forms from Jacobi Forms

For any polynomial $P \in \mathbb{C}[z_{11}, \cdots, z_{mn}]$ with $Z = (z_{kj}) \in \mathbb{C}^{(m,n)}$, we put

$$P(\partial_Z) := P\left(\frac{\partial}{\partial z_{11}}, \cdots, \frac{\partial}{\partial z_{mn}}\right).$$

Theorem (J.-H. Yang: 1995)

Let P be a *homogeneous pluriharmonic* polynomial in $\mathbb{C}[z_{11}, \cdots, z_{mn}]$. Let $f \in J_{\rho,\mathcal{M}}(\Gamma_n)$. Then

$$P(\partial_Z) f(\Omega, Z)\big|_{Z=0}$$

is a vector valued modular form with respect to a new rational representation of $GL(n, \mathbb{C})$.

Remarks : (a) D. Mumford, M. Nori and P. Norman (1991: Tata Lecture III) proved the similar result for theta functions. The above theorem is a generalization of their result because theta functions are special Examples of Jacobi forms.

(b) Applying the ideas of the proof of the Above theorem to Eisenstein series and theta functions, I obtained interesting identities.

11. Maass-Jacobi Forms

Definition. Let

$$\Gamma_{n,m} := Sp(n,\mathbb{Z}) \ltimes H_{\mathbb{Z}}^{(n,m)}$$

be the discrete subgroup of G^J, where

$$H_{\mathbb{Z}}^{(n,m)} = \left\{ (\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(n,m)} \mid \lambda, \mu, \kappa \text{ are integral} \right\}.$$

A smooth function $f : \mathbb{H}_{n,m} \longrightarrow \mathbb{C}$ is called a Maass-Jacobi form on $\mathbb{H}_{n,m}$ if f satisfies the following conditions (MJ1)-(MJ3) :

(MJ1) f is invariant under $\Gamma_{n,m}$.

(MJ2) f is an eigenfunction of the Laplacian $\Delta_{n,m;A,B}$.

(MJ3) f has a polynomial growth, that is, there exist a constant $C > 0$ and a positive integer N such that

$$|f(X+iY,Z)| \leq C\,|p(Y)|^N \quad \text{as } \det Y \longrightarrow \infty,$$

where $p(Y)$ is a polynomial in $Y = (y_{ij})$.

Remark: Let $\mathbb{D}_*$ be a commutative subalgebra of $\mathbb{D}(\mathbb{H}_{n,m})$ containing the Laplacian $\Delta_{n,m;A,B}$. We say that a smooth function $f : \mathbb{H}_{n,m} \longrightarrow \mathbb{C}$ is a Maass-Jacobi form w.r.t $\mathbb{D}_*$ if f satisfies the conditions $(MJ1)$, $(MJ2)_*$ and $(MJ3)$: the condition $(MJ2)_*$ is given by

$(MJ2)_*$ f **is an eigenfunction of any operator in** $\mathbb{D}_*$.

It is natural to propose the following problems.

Problem A : Find all the eigenfunctions of $\Delta_{n,m;A,B}$.

Problem B : Construct Maass-Jacobi forms.

If we find a *nice* eigenfunction ϕ of $\Delta_{n,m;A,B}$, we can construct a Maass-Jacobi form f_ϕ on $\mathbb{H}_{n,m}$ in the usual way defined by

$$f_\phi(\Omega, Z) := \sum_{\gamma \in \Gamma^\infty_{n,m} \backslash \Gamma_{n,m}} \phi\big(\gamma \cdot (\Omega, Z)\big),$$

where

$$\Gamma^\infty_{n,m} = \left\{ \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, (\lambda, \mu; \kappa) \right) \in \Gamma_{n,m} \,\middle|\, C = 0 \right\}$$

is a subgroup of $\Gamma_{n,m}$.

We give some examples of eigenfunctions of $\Delta_{1,1;1,1}$.

(1) $h(x,y) = y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|a|y)\, e^{2\pi i a x}$ $(s \in \mathbb{C},\ a \neq 0)$ with eigenvalue $s(s-1)$. Here

$$K_s(z) := \frac{1}{2}\int_0^\infty \exp\left\{-\frac{z}{2}(t+t^{-1})\right\} t^{s-1}\, dt,$$

where $\mathrm{Re}\, z > 0$.

(2) $y^s,\ y^s x,\ y^s u$ $(s \in \mathbb{C})$ with eigenvalue $s(s-1)$.

(3) $y^s v,\ y^s uv,\ y^s xv$ with eigenvalue $s(s+1)$.

(4) $x,\ y,\ u,\ v,\ xv,\ uv$ with eigenvalue 0.

(5) All Maass wave forms.

We define $\mathbb{D}_{\rho,\mathcal{M}}$ to be the algebra of all differential operators D on $\mathbb{H}_{n,m}$ satisfying the following condition

$$(Df)|_{\rho,\mathcal{M}}[g] = D(f|_{\rho,\mathcal{M}}[g])$$

for all $f \in C^\infty(\mathbb{H}_{n,m}, V_\rho)$ and for all $g \in G^J$.

We denote by $\mathbb{Z}_{\rho,\mathcal{M}}$ the center of $\mathbb{D}_{\rho,\mathcal{M}}$.

We define another notion of Maass-Jacobi forms as follows.

Definition. A vector-valued smooth function $\phi : \mathbb{H}_{n,m} \longrightarrow V_\rho$ is called a Maass-Jacobi form on $\mathbb{H}_{n,m}$ of type ρ and index $\mathcal{M}$ if it satisfies the following conditions $(MJ1)_{\rho,\mathcal{M}}$, $(MJ2)_{\rho,\mathcal{M}}$ and$(MJ3)_{\rho,\mathcal{M}}$:

$(MJ1)_{\rho,\mathcal{M}}$ $\quad \phi|_{\rho,\mathcal{M}}[\gamma] = \phi$ for all $\gamma \in \Gamma_{n,m}$.

$(MJ2)_{\rho,\mathcal{M}}$ $\quad f$ is an eigenfunction of all operators in $\mathbb{Z}_{\rho,\mathcal{M}}$

$(MJ3)_{\rho,\mathcal{M}}$ $\quad f$ has a growth condition

$$\phi(\Omega, Z) = O\left(e^{a \det Y} \cdot e^{2\pi \operatorname{tr}(\mathcal{M}[V]Y^{-1})}\right)$$

as $\det Y \longrightarrow \infty$ for some $a > 0$.

Remarks : The case when $n = 1$, $m = 1$ and $\rho = \det^k$ $(k = 0, 1, 2, \cdots)$ was studied by R. Berndt and R. Schmidt, A. Pitale and K. Bringmann and O. Richter. They proved that the center $\mathcal{Z}_{\det^k, \mathcal{M}}$ of $\mathbb{D}_{\det^k, \mathcal{M}}$ is the polynomial algebra with one generator $\mathcal{C}^{k, \mathcal{M}}$ of degree three.

12. The Schrödinger-Weil Representation

Let $\mathcal{M}$ be a positive symmetric real $m \times m$ matrix. We can construct the so-called **Schrödinger-Weil representation** $\omega_{\mathcal{M}}$ of the Jacobi group G^J.

There are some applications of the Schrödinger-Weil representation $\omega_{\mathcal{M}}$:

(1) Construction of Jacobi forms

(2) Theta Sums

(3) Unitary Representations of G^J or G_2^J

(4) Maass-Jacobi Forms

The details can be found in the following references:

[1] K. Takase, On Unitary Representations of the Jacobi Group, J. reine angew. Math. **430** (1992), 130-149.

[2] J.-H. Yang, The Weil representations of the Jacobi group, Proceedings of the international conference on Geometry, Number Theory and Representation, Kyung Moon Sa (2013), 169-204.

[3] J.-H. Yang, A Note on Maass-Jacobi forms II, Kyungpook Math. J. (2013), 49-86.

[4] J.-H. Yang, The Schrödinger-Weil Representation and Theta Sums, submitted (2013).

13. Open Problems

[1] Analogue of Hirzebruch-Mumford Proportionality Th:

[Hirzebruch-Mumford] Assume

- $E_0 = G \times_K \mathbb{C}^r, \quad G = Sp(n, \mathbb{R}),\ K = U(n),\ \tau : K \longrightarrow GL(r, \mathbb{C})$
- Γ is an arithmetic subgroup of $Sp(n, \mathbb{Z})$
- E is a hol VB on $\mathcal{A}_{n,\Gamma} := \Gamma \backslash G/K$
- E_0 carries a G-equivariant Hermitian metric h_0 which induces a Hermitian metric h on E.

Then there exists a hol VB $\widetilde{E}$ on a toroidal compactification $\tilde{\mathcal{A}}_{n,\Gamma}$ of $\mathcal{A}_{n,\Gamma}$ and a natural metric on $\mathbb{H}_n = G/K$ such that

$$c^{\alpha}\big(\widetilde{E}\big) = (-1)^{\frac{1}{2}n(n+1)}\,\mathrm{vol}\,(\Gamma \backslash \mathbb{H}_n)\ c^{\alpha}\big(\check{E}\big).$$

Here $\alpha = (\alpha_1, \cdots, \alpha_r) \in \mathbb{Z}^r_{\geq 0}$ and $\check{E}$ is a $G_{\mathbb{C}}$-equivariant hol VB on the compact dual of $\mathbb{H}_n$.

[2] Compute the cohomology $H^*(\mathcal{A}_{n,m,\Gamma}, \bullet)$ of $\mathcal{A}_{n,m,\Gamma}$. Investigate the intersection cohomology of $\mathcal{A}_{n,m,\Gamma}$.

[3] Generalize the trace formula of Sophie Moreal on the Siegel modular variety to the universal abelian variety.

Let S^K be the Siegel modular variety associated to $\mathbf{GSp}_{2n}$ and let $K \subset \mathbf{G}(\mathbb{A}_f)$ be an compact open subgroup. Let V be an algebraic repn of G. Let p be a prime such that $K = \mathbf{G}(\mathbb{Z}_p)K^p$ with $K^p \subset \mathbf{G}(\mathbb{A}_f^p)$. Then for all $f \in C_c^\infty(K \backslash \mathbf{G}(\mathbb{A}_f)/K)$ such that $f^\infty = f^{\infty,p}\mathbf{1}_{\mathbf{G}(\mathbb{Z}_p)}$ and for any suff large integer j, we have

$$\mathrm{Tr}(\mathrm{Frob}_p^j \times f^\infty, \mathrm{IH}^*(S^K, V)) = \sum_{(\mathbf{H},s,\eta_0)} \iota(\mathbf{G},\mathbf{H})\, ST^H\big(f_{\mathbf{H}}^{(j)}\big).$$

Here $(\mathbf{H}, s, \eta_0)$ is the equivalence class of triples of elliptic endoscopies of G such that $\mathbf{H}_{\mathbb{R}}$ admits an elliptic maximal torus, $f_{\mathbf{H}}^{(j)}$ is a function on $\mathbf{H}(\mathbb{A})$ and ST^H is the geometric side of the stable trace formula on $\mathbf{H}$.

[4] Develop the theory of the stability of Jacobi forms.

[5] Compute geodesics, distances, scalar curvatures, Ricci curvatures, Chern classes and so on.
Express the center of $\mathscr{U}(\mathfrak{g}^J_{\mathbb{C}})$ explicitly.
Compute the center of $\mathbb{D}(\mathbb{H}_{n,m})$.

[6] Develop the spectral theory of $\Delta_{n,m;A,B}$ on $\Gamma\backslash\mathbb{H}_{n,m}$ for an arithmetic subgroup Γ of $\Gamma_{n,m}$.

[7] Develop the theory of harmonic analysis on the Siegel-Jacobi disk $\mathbb{D}_{n,m}$.

[8] Study unitary representations of G^J, equivalently, $Sp(n,\mathbb{R})$ (orbit method for G^J).

[9] Attach Galois representations to cuspidal Jacobi forms.

[10] Automorphic L-functions for G^J.

[11] Trace formula for G^J.

[12] Decompose $L^2\big(G^J(\mathbb{Q}\backslash G^J(\mathbb{A})\big)$ into irreducibles explicitly.

[13] Analogue of Langlands program (Differences : non-symmetric space, non-commutative Hecke algebra, no root systems, no multiplicity one theorem,$\cdots$).

[14] **Construct Maass-Jacobi forms. Express the Fourier expansion of a Maass-Jacobi form explicitly.**

[15] **Investigate the relations among Jacobi forms, hyperbolic Kac-Moody algebras, infinite products, the Moonshine and the monster group.**

[16] **Study complete mixed mock Jacobi forms and skew-holomorphic Jacobi forms relating to mock theta functions, Appell functions (S. Zwegers), mixed mock modular forms (Don Zagier).**

[17] **Applications to physics (quantum mechanics, quantum optics, coherent states, $\cdots$), elliptic genera, singularity theory of K. Saito, $\cdots$.**

Many Thanks !!!