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A NOTE ON A FUNDAMENTAL DOMAIN FOR SIEGEL-JACOBI SPACE

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Abstract. In this paper, we study a fundamental domain for the Siegel-Jacobi space $Sp(g,\mathbb{Z}) \ltimes H_{\mathbb{Z}}^{(g,h)} \backslash \mathbb{H}_g \times \mathbb{C}^{(h,g)}$.

1. Introduction

For a given fixed positive integer g, we let

$$\mathbb{H}_g = \{\, \Omega \in \mathbb{C}^{(g,g)} \mid \Omega = {}^t\Omega, \quad \operatorname{Im}\Omega > 0 \,\}$$

be the Siegel upper half plane of degree g and let

$$Sp(g,\mathbb{R}) = \{ M \in \mathbb{R}^{(2g,2g)} \mid {}^tMJ_gM = J_g \,\}$$

be the symplectic group of degree g, where $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F for two positive integers k and l, ${}^tM$ denotes the transpose matrix of a matrix M and

$$J_g = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}.$$

$Sp(g,\mathbb{R})$ acts on $\mathbb{H}_g$ transitively by

$$M \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1}, \tag{1.1}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g,\mathbb{R})$ and $\Omega \in \mathbb{H}_g$. Let Γ_g be the Siegel modular group of degree g. C. L. Siegel [8] found a fundamental domain $\mathcal{F}_g$ for $\Gamma_g \backslash \mathbb{H}_g$ and calculated the volume of $\mathcal{F}_g$. We also refer to [2], [4], [10] for some details on $\mathcal{F}_g$.

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For two positive integers g and h, we consider the Heisenberg group

$$H_{\mathbb{R}}^{(g,h)} = \{ (\lambda, \mu; \kappa) \mid \lambda, \mu \in \mathbb{R}^{(h,g)},\ \kappa \in \mathbb{R}^{(h,h)},\ \kappa + \mu\, {}^t\lambda \text{ symmetric } \}$$

endowed with the following multiplication law

$$(\lambda, \mu; \kappa) \circ (\lambda', \mu'; \kappa') = (\lambda + \lambda', \mu + \mu'; \kappa + \kappa' + \lambda\, {}^t\mu' - \mu\, {}^t\lambda').$$

We define the semidirect product of $Sp(g, \mathbb{R})$ and $H_{\mathbb{R}}^{(g,h)}$

$$G^J = Sp(g, \mathbb{R}) \ltimes H_{\mathbb{R}}^{(g,h)}$$

endowed with the following multiplication law

$$(M, (\lambda, \mu; \kappa)) \cdot (M', (\lambda', \mu'; \kappa')) = \, (MM', (\tilde{\lambda} + \lambda', \tilde{\mu} + \mu'; \kappa + \kappa' + \tilde{\lambda}\, {}^t\mu' - \tilde{\mu}\, {}^t\lambda'))$$

with $M, M' \in Sp(g, \mathbb{R}), (\lambda, \mu; \kappa),\ (\lambda', \mu'; \kappa') \in H_{\mathbb{R}}^{(g,h)}$ and $(\tilde{\lambda}, \tilde{\mu}) = (\lambda, \mu)M'$. Then G^J acts on $\mathbb{H}_g \times \mathbb{C}^{(h,g)}$ transitively by

$$(1.2) \qquad (M, (\lambda, \mu; \kappa)) \cdot (\Omega, Z) = (M \cdot \Omega, (Z + \lambda\Omega + \mu)(C\Omega + D)^{-1}),$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g, \mathbb{R}),\ (\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(g,h)}$ and $(\Omega, Z) \in \mathbb{H}_g \times \mathbb{C}^{(h,g)}$. We note that the Jacobi group G^J is *not* a reductive Lie group and also that the space $\mathbb{H}_g \times \mathbb{C}^{(h,g)}$ is not a symmetric space. We refer to [11]-[14] and [16] about automorphic forms on G^J and topics related to the content of this paper. From now on, we write $\mathbb{H}_{g,h} := \mathbb{H}_g \times \mathbb{C}^{(h,g)}$.

We let

$$\Gamma_{g,h} := \Gamma_g \ltimes H_{\mathbb{Z}}^{(g,h)}$$

be the discrete subgroup of G^J, where

$$H_{\mathbb{Z}}^{(g,h)} = \{ (\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(g,h)} \mid \lambda, \mu \in \mathbb{Z}^{(h,g)},\ \ \kappa \in \mathbb{Z}^{(h,h)} \}.$$

The aim of this paper is to find a fundamental domain for $\Gamma_{g,h} \backslash \mathbb{H}_{g,h}$. This article is organized as follows. In Section 2, we review the Minkowski domain and the Siegel's fundamental domain $\mathcal{F}_g$ roughly. In Section 3, we find a fundamental domain for $\Gamma_{g,h} \backslash \mathbb{H}_{g,h}$ and present Riemannian metrics on the fundamental domain invariant under the action (1.2) of the Jacobi group G^J. In Section 4, we investigate the spectral theory of the Laplacian on the abelian variety A_Ω associated to $\Omega \in \mathcal{F}_g$.

2. Review on a Fundamental Domain $\mathcal{F}_g$ for $\Gamma_g\backslash\mathbb{H}_g$

We let

$$\mathcal{P}_g = \left\{ Y \in \mathbb{R}^{(g,g)} \mid Y = {}^tY > 0 \right\}$$

be an open cone in $\mathbb{R}^N$ with $N = g(g+1)/2$. The general linear group $GL(g,\mathbb{R})$ acts on $\mathcal{P}_g$ transitively by

$$g \circ Y := gY\,{}^tg, \qquad g \in GL(g,\mathbb{R}),\ Y \in \mathcal{P}_g. \tag{2.1}$$

Thus $\mathcal{P}_g$ is a symmetric space diffeomorphic to $GL(g,\mathbb{R})/O(g)$. For a matrix $A \in F^{(k,l)}$ and $B \in F^{(k,l)}$, we write $A[B] = {}^tBAB$ and for a square matrix A, $\sigma(A)$ denotes the trace of A.

The fundamental domain $\mathcal{R}_g$ for $GL(g,\mathbb{Z})\backslash\mathcal{P}_g$ which was found by H. Minkowski [5] is defined as a subset of $\mathcal{P}_g$ consisting of $Y = (y_{ij}) \in \mathcal{P}_g$ satisfying the following conditions (M.1)-(M.2) (cf. [2, p. 191] or [4, p. 123]):

(M.1) $aY\,{}^ta \geq y_{kk}$ for every $a = (a_i) \in \mathbb{Z}^g$ in which $a_k, \cdots, a_g$ are relatively prime for $k = 1, 2, \cdots, g$.

(M.2) $y_{k,k+1} \geq 0$ for $k = 1, \cdots, g-1$.

We say that a point of $\mathcal{R}_g$ is *Minkowski reduced* or simply *M-reduced*. $\mathcal{R}_g$ has the following properties (R1)-(R6):

(R1) For any $Y \in \mathcal{P}_g$, there exist a matrix $A \in GL(g,\mathbb{Z})$ and $R \in \mathcal{R}_g$ such that $Y = R[A]$ (cf. [2, p. 191] or [4, p. 139]). That is,

$$GL(g,\mathbb{Z}) \circ \mathcal{R}_g = \mathcal{P}_g.$$

(R2) $\mathcal{R}_g$ is a convex cone through the origin bounded by a finite number of hyperplanes. $\mathcal{R}_g$ is closed in $\mathcal{P}_g$ (cf. [4, p. 139]).

(R3) If Y and $Y[A]$ lie in $\mathcal{R}_g$ for $A \in GL(g,\mathbb{Z})$ with $A \neq \pm I_g$, then Y lies on the boundary $\partial\mathcal{R}_g$ of $\mathcal{R}_g$. Moreover $\mathcal{R}_g \cap (\mathcal{R}_g[A]) \neq \emptyset$ for only finitely many $A \in GL(g,\mathbb{Z})$ (cf. [4, p. 139]).

(R4) If $Y = (y_{ij})$ is an element of $\mathcal{R}_g$, then

$$y_{11} \leq y_{22} \leq \cdots \leq y_{gg} \quad \text{and} \quad |y_{ij}| < \frac{1}{2} y_{ii} \quad \text{for } 1 \leq i < j \leq g.$$

We refer to [2, p. 192] or [4, pp. 123-124].

Remark. Grenier [1] found another fundamental domain for $GL(g,\mathbb{Z})\backslash\mathcal{P}_g$.

For $Y=(y_{ij})\in\mathcal{P}_g$, we put

$$dY=(dy_{ij})\qquad\text{and}\qquad \frac{\partial}{\partial Y}=\left(\frac{1+\delta_{ij}}{2}\,\frac{\partial}{\partial y_{ij}}\right).$$

Then we can see easily that

$$ds^2=\sigma((Y^{-1}dY)^2) \tag{2.2}$$

is a $GL(g,\mathbb{R})$-invariant Riemannian metric on $\mathcal{P}_g$ and its Laplacian is given by

$$\Delta=\sigma\left(\left(Y\frac{\partial}{\partial Y}\right)^2\right).$$

We also can see that

$$d\mu_g(Y)=(\det Y)^{-\frac{g+1}{2}}\prod_{i\le j}dy_{ij}$$

is a $GL(g,\mathbb{R})$-invariant volume element on $\mathcal{P}_g$. The metric ds^2 on $\mathcal{P}_g$ induces the metric $ds^2_{\mathcal{R}}$ on $\mathcal{R}_g$. Minkowski [5] calculated the volume of $\mathcal{R}_g$ for the volume element $[dY]:=\prod_{i\le j}dy_{ij}$ explicitly. Later Siegel [7], [9] computed the volume of $\mathcal{R}_g$ for the volume element $[dY]$ by a simple analytic method and generalized this case to the case of any algebraic number field.

Siegel [8] determined a fundamental domain $\mathcal{F}_g$ for $\Gamma_g\backslash\mathbb{H}_g$. We say that $\Omega=X+iY\in\mathbb{H}_g$ with X,Y real is *Siegel reduced* or *S-reduced* if it has the following three properties:

(S.1) $\det(\mathrm{Im}\,(\gamma\cdot\Omega))\le\det(\mathrm{Im}\,(\Omega))\qquad$ for all $\gamma\in\Gamma_g$;

(S.2) $Y=\mathrm{Im}\,\Omega$ is M-reduced, that is, $Y\in\mathcal{R}_g$;

(S.3) $|x_{ij}|\le\frac{1}{2}\quad$ for $1\le i,j\le g,$ where $X=(x_{ij})$.

$\mathcal{F}_g$ is defined as the set of all Siegel reduced points in $\mathbb{H}_g$. Using the highest point method, Siegel proved the following (F1)-(F3) (cf. [2, pp. 194-197] or [4, p. 169]):

(F1) $\Gamma_g\cdot\mathcal{F}_g=\mathbb{H}_g$, i.e., $\mathbb{H}_g=\cup_{\gamma\in\Gamma_g}\gamma\cdot\mathcal{F}_g$.

(F2) $\mathcal{F}_g$ is closed in $\mathbb{H}_g$.

(F3) $\mathcal{F}_g$ is connected and the boundary of $\mathcal{F}_g$ consists of a finite number of hyperplanes.

For $\Omega=(\omega_{ij})\in\mathbb{H}_g$, we write $\Omega=X+iY$ with $X=(x_{ij}),\ Y=(y_{ij})$ real and $d\Omega=(d\omega_{ij})$. We also put

$$\frac{\partial}{\partial\Omega}=\left(\frac{1+\delta_{ij}}{2}\,\frac{\partial}{\partial\omega_{ij}}\right)\qquad\text{and}\qquad\frac{\partial}{\partial\overline{\Omega}}=\left(\frac{1+\delta_{ij}}{2}\,\frac{\partial}{\partial\overline{\omega}_{ij}}\right).$$

Then

$$ds_*^2 = \sigma(Y^{-1}d\Omega\, Y^{-1}d\overline{\Omega}) \tag{2.3}$$

is a $Sp(g,\mathbb{R})$-invariant Kähler metric on $\mathbb{H}_g$ (cf. [8]) and H. Maass [3] proved that its Laplacian is given by

$$\Delta_* = 4\,\sigma\left(Y\ {}^t\!\left(Y\frac{\partial}{\partial\overline{\Omega}}\right)\frac{\partial}{\partial\Omega}\right). \tag{2.4}$$

And

$$dv_g(\Omega) = (\det Y)^{-(g+1)} \prod_{1\le i\le j\le g} dx_{ij} \prod_{1\le i\le j\le g} dy_{ij} \tag{2.5}$$

is a $Sp(g,\mathbb{R})$-invariant volume element on $\mathbb{H}_g$ (cf. [10, p. 130]). The metric ds_*^2 given by (2.3) induces a metric $ds_{\mathcal{F}}^2$ on $\mathcal{F}_g$.

Siegel [8] computed the volume of $\mathcal{F}_g$

$$\mathrm{vol}\,(\mathcal{F}_g) = 2\prod_{k=1}^{g} \pi^{-k}\Gamma(k)\zeta(2k), \tag{2.6}$$

where $\Gamma(s)$ denotes the Gamma function and $\zeta(s)$ denotes the Riemann zeta function. For instance,

$$\mathrm{vol}\,(\mathcal{F}_1) = \frac{\pi}{3}, \quad \mathrm{vol}\,(\mathcal{F}_2) = \frac{\pi^3}{270}, \quad \mathrm{vol}\,(\mathcal{F}_3) = \frac{\pi^6}{127575}, \quad \mathrm{vol}\,(\mathcal{F}_4) = \frac{\pi^{10}}{200930625}.$$

3. A Fundamental Domain for $\Gamma_{g,h}\backslash\mathbb{H}_{g,h}$

Let E_{kj} be the $h\times g$ matrix with entry 1 where the k-th row and the j-th column meet, and all other entries 0. For an element $\Omega\in\mathbb{H}_g$, we set for brevity

$$F_{kj}(\Omega) := E_{kj}\Omega, \qquad 1\le k\le h,\ 1\le j\le g. \tag{3.1}$$

For each $\Omega\in\mathcal{F}_g$, we define a subset P_Ω of $\mathbb{C}^{(h,g)}$ by

$$P_\Omega = \left\{ \sum_{k=1}^{h}\sum_{j=1}^{g}\lambda_{kj}E_{kj} + \sum_{k=1}^{h}\sum_{j=1}^{g}\mu_{kj}F_{kj}(\Omega)\ \Big|\ 0\le\lambda_{kj},\mu_{kj}\le 1 \right\}.$$

For each $\Omega\in\mathcal{F}_g$, we define the subset D_Ω of $\mathbb{H}_{g,h}$ by

$$D_\Omega := \{\,(\Omega,Z)\in\mathbb{H}_{g,h}\mid Z\in P_\Omega\,\}.$$

We define

$$\mathcal{F}_{g,h} := \cup_{\Omega\in\mathcal{F}_g} D_\Omega.$$

Theorem 3.1. *$\mathcal{F}_{g,h}$ is a fundamental domain for $\Gamma_{g,h}\backslash\mathbb{H}_{g,h}$.*

PROOF. Let $(\tilde{\Omega},\tilde{Z})$ be an arbitrary element of $\mathbb{H}_{g,h}$. We must find an element (Ω, Z) of $\mathcal{F}_{g,h}$ and an element $\gamma^J=(\gamma,(\lambda,\mu;\kappa))\in\Gamma_{g,h}$ with $\gamma\in\Gamma_g$ such that $\gamma^J\cdot(\Omega,Z)=(\tilde{\Omega},\tilde{Z})$. Since $\mathcal{F}_g$ is a fundamental domain for $\Gamma_g\backslash\mathbb{H}_g$, there exists an element γ of Γ_g and an element Ω of $\mathcal{F}_g$ such that $\gamma\cdot\Omega=\tilde{\Omega}$. Here Ω is unique up to the boundary of $\mathcal{F}_g$.

We write

$$\gamma=\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in\Gamma_g.$$

It is easy to see that we can find $\lambda,\mu\in\mathbb{Z}^{(h,g)}$ and $Z\in P_\Omega$ satisfying the equation

$$Z+\lambda\Omega+\mu=\tilde{Z}(C\Omega+D).$$

If we take $\gamma^J=(\gamma,(\lambda,\mu;0))\in\Gamma_{g,h}$, we see that $\gamma^J\cdot(\Omega,Z)=(\tilde{\Omega},\tilde{Z})$. Therefore we obtain

$$\mathbb{H}_{g,h}=\cup_{\gamma^J\in\Gamma_{g,h}}\gamma^J\cdot\mathcal{F}_{g,h}.$$

Let (Ω,Z) and $\gamma^J\cdot(\Omega,Z)$ be two elements of $\mathcal{F}_{g,h}$ with $\gamma^J=(\gamma,(\lambda,\mu;\kappa))\in\Gamma_{g,h}$. Then both Ω and $\gamma\cdot\Omega$ lie in $\mathcal{F}_g$. Therefore both of them either lie in the boundary of $\mathcal{F}_g$ or $\gamma=\pm I_{2g}$. In the case that both Ω and $\gamma\cdot\Omega$ lie in the boundary of $\mathcal{F}_g$, both (Ω,Z) and $\gamma^J\cdot(\Omega,Z)$ lie in the boundary of $\mathcal{F}_{g,h}$. If $\gamma=\pm I_{2g}$, we have

$$Z\in P_\Omega \quad \text{and} \quad \pm\,(Z+\lambda\Omega+\mu)\in P_\Omega,\quad \lambda,\mu\in\mathbb{Z}^{(h,g)}. \tag{3.2}$$

From the definition of P_Ω and (3.2), we see that either $\lambda=\mu=0,\ \gamma\neq -I_{2g}$ or both Z and $\pm(Z+\lambda\Omega+\mu)$ lie on the boundary of the parallelepiped P_Ω. Hence either both(Ω,Z) and $\gamma^J\cdot(\Omega,Z)$ lie in the boundary of $\mathcal{F}_{g,h}$ or $\gamma^J=(I_{2g},(0,0;\kappa))\in\Gamma_{g,h}$. Consequently $\mathcal{F}_{g,h}$ is a fundamental domain for $\Gamma_{g,h}\backslash\mathbb{H}_{g,h}$. □

For a coordinate $(\Omega,Z)\in\mathbb{H}_{g,h}$ with $\Omega=(\omega_{\mu\nu})\in\mathbb{H}_g$ and $Z=(z_{kl})\in\mathbb{C}^{(h,g)}$, we put

$$\begin{aligned}
\Omega &= X+iY, \quad X=(x_{\mu\nu}), \quad Y=(y_{\mu\nu}) \ \text{ real},\\
Z &= U+iV, \quad U=(u_{kl}), \quad V=(v_{kl}) \ \text{ real},\\
d\Omega &= (d\omega_{\mu\nu}), \quad dX=(dx_{\mu\nu}), \quad dY=(dy_{\mu\nu}),\\
dZ &= (dz_{kl}), \quad dU=(du_{kl}), \quad dV=(dv_{kl}),\\
d\overline{\Omega} &= (d\overline{\omega}_{\mu\nu}), \quad d\overline{Z}=(d\bar{z}_{kl}),
\end{aligned}$$

$$\frac{\partial}{\partial\Omega} = \left(\frac{1+\delta_{\mu\nu}}{2}\,\frac{\partial}{\partial\omega_{\mu\nu}} \right), \quad \frac{\partial}{\partial\overline{\Omega}} = \left(\frac{1+\delta_{\mu\nu}}{2}\,\frac{\partial}{\partial\overline{\omega}_{\mu\nu}} \right),$$

$$\frac{\partial}{\partial Z} = \begin{pmatrix} \frac{\partial}{\partial z_{11}} & \cdots & \frac{\partial}{\partial z_{h1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial z_{1g}} & \cdots & \frac{\partial}{\partial z_{hg}} \end{pmatrix}, \quad \frac{\partial}{\partial\overline{Z}} = \begin{pmatrix} \frac{\partial}{\partial\overline{z}_{11}} & \cdots & \frac{\partial}{\partial\overline{z}_{h1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial\overline{z}_{1g}} & \cdots & \frac{\partial}{\partial\overline{z}_{hg}} \end{pmatrix}.$$

Remark. The following metric

$$\begin{aligned} ds^2_{g,h} =& \,\sigma\left(Y^{-1}d\Omega\, Y^{-1}d\overline{\Omega}\right) + \sigma\left(Y^{-1}\,{}^tV\,V\,Y^{-1}d\Omega\,Y^{-1}d\overline{\Omega}\right) \\ &+ \sigma\left(Y^{-1}\,{}^t(dZ)\,d\overline{Z}\right) \\ &- \sigma\left(\,V\,Y^{-1}d\Omega\,Y^{-1}\,{}^t(d\overline{\Omega}) + V\,Y^{-1}d\overline{\Omega}\,Y^{-1}\,{}^t(dZ)\,\right) \end{aligned}$$

is a Kähler metric on $\mathbb{H}_{g,h}$ which is invariant under the action (1.2) of the Jacobi group G^J. Its Laplacian is given by

$$\begin{aligned} \Delta_{g,h} =& \,4\,\sigma\left(Y\ {}^t\!\left(Y\frac{\partial}{\partial\overline{\Omega}}\right)\frac{\partial}{\partial\Omega}\right) + 4\,\sigma\left(Y\,\frac{\partial}{\partial Z}\ {}^t\!\left(\frac{\partial}{\partial\overline{Z}}\right)\right) \\ &+ 4\,\sigma\left(VY^{-1}\,{}^tV\ {}^t\!\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial Z}\right) \\ &+ 4\,\sigma\left(V\ {}^t\!\left(Y\frac{\partial}{\partial\overline{\Omega}}\right)\frac{\partial}{\partial Z}\right) + 4\sigma\left({}^tV\ {}^t\!\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial\Omega}\right). \end{aligned}$$

The following differential form

$$dv_{g,h} = \left(\det Y\right)^{-(g+h+1)}[dX]\wedge[dY]\wedge[dU]\wedge[dV]$$

is a G^J-invariant volume element on $\mathbb{H}_{g,h}$, where

$$[dX] = \wedge_{\mu\le\nu}dx_{\mu\nu}, \quad [dY] = \wedge_{\mu\le\nu}dy_{\mu\nu}, \quad [dU] = \wedge_{k,l}du_{kl} \quad \text{and} \quad [dV] = \wedge_{k,l}dv_{kl}.$$

The point is that the invariant metric $ds^2_{g,h}$ and its Laplacian are beautifully expressed in terms of the *trace* form. The proof of the above facts can be found in [15].

4. Spectral Decomposition of $L^2(A_\Omega)$

We fix two positive integers g and h throughout this section.

For an element $\Omega\in\mathbb{H}_g$, we set

$$L_\Omega := \mathbb{Z}^{(h,g)} + \mathbb{Z}^{(h,g)}\Omega$$

We use the notation (3.1). It follows from the positivity of $\operatorname{Im}\Omega$ that the elements $E_{kj},\, F_{kj}(\Omega)\,(1\le k\le h,\ 1\le j\le g)$ of L_Ω are linearly independent over $\mathbb{R}$. Therefore L_Ω is a lattice in $\mathbb{C}^{(h,g)}$ and the set $\{\,E_{kj},\, F_{kj}(\Omega)\,|\, 1\le k\le h,\ 1\le j\le g\,\}$ forms an integral basis of L_Ω. We see easily that if Ω is an element of $\mathbb{H}_g$, the period matrix $\Omega_* := (I_g,\Omega)$ satisfies the Riemann conditions (RC.1) and (RC.2) :

(RC.1) $\quad \Omega_* J_g\, {}^t\Omega_* = 0\,;$

(RC.2) $\quad -\frac{1}{i}\Omega_* J_g\, {}^t\overline{\Omega}_* > 0.$

Thus the complex torus $A_\Omega := \mathbb{C}^{(h,g)}/L_\Omega$ is an abelian variety. For more details on A_Ω, we refer to [2] and [6].

It might be interesting to investigate the spectral theory of the Laplacian $\Delta_{g,h}$ on a fundamental domain $\mathcal{F}_{g,h}$. But this work is very complicated and difficult at this moment. It may be that the first step is to develop the spectral theory of the Laplacian Δ_Ω on the abelian variety A_Ω. The second step will be to study the spectral theory of the Laplacian Δ_* (see (2.4)) on the moduli space $\Gamma_g\backslash\mathbb{H}_g$ of principally polarized abelian varieties of dimension g. The final step would be to combine the above steps and more works to develop the spectral theory of the Laplacian $\Delta_{g,h}$ on $\mathcal{F}_{g,h}$. In this section, we deal only with the spectral theory of Δ_Ω on $L^2(A_\Omega)$.

We fix an element $\Omega = X+iY$ of $\mathbb{H}_g$ with $X=\operatorname{Re}\Omega$ and $Y=\operatorname{Im}\Omega$. For a pair (A,B) with $A,B\in\mathbb{Z}^{(h,g)}$, we define the function $E_{\Omega;A,B} : \mathbb{C}^{(h,g)} \longrightarrow \mathbb{C}$ by

$$E_{\Omega;A,B}(Z) = e^{2\pi i\left(\sigma\,({}^tAU\,)+\,\sigma\,((B-AX)Y^{-1}\,{}^tV)\right)},$$

where $Z=U+iV$ is a variable in $\mathbb{C}^{(h,g)}$ with real U,V.

Lemma 4.1. *For any $A,B\in\mathbb{Z}^{(h,g)}$, the function $E_{\Omega;A,B}$ satisfies the following functional equation*

$$E_{\Omega;A,B}(Z+\lambda\Omega+\mu) = E_{\Omega;A,B}(Z),\quad Z\in\mathbb{C}^{(h,g)}$$

for all $\lambda,\mu\in\mathbb{Z}^{(h,g)}$. Thus $E_{\Omega;A,B}$ can be regarded as a function on A_Ω.

Proof. We write $\Omega = X + iY$ with real X, Y. For any $\lambda, \mu \in \mathbb{Z}^{(h,g)}$, we have

$$\begin{aligned} E_{\Omega;A,B}(Z+\lambda\Omega+\mu) &= E_{\Omega;A,B}((U+\lambda X+\mu)+i(V+\lambda Y)) \\ &= e^{2\pi i\left\{\,\sigma\,({}^tA(U+\lambda X+\mu))+\sigma\,((B-AX)Y^{-1}\,{}^t(V+\lambda Y))\,\right\}} \\ &= e^{2\pi i\left\{\,\sigma\,({}^tAU+{}^tA\lambda X+{}^tA\mu)+\sigma\,((B-AX)Y^{-1}\,{}^tV+B\,{}^t\lambda-AX\,{}^t\lambda)\right\}} \\ &= e^{2\pi i\left\{\,\sigma\,({}^tAU)+\sigma\,((B-AX)Y^{-1}\,{}^tV)\right\}} \\ &= E_{\Omega;A,B}(Z). \end{aligned}$$

Here we used the fact that ${}^tA\mu$ and $B\,{}^t\lambda$ are integral. □

We use the notations in Section 3.

Lemma 4.2. *The metric*

$$ds^2_\Omega = \sigma\left((Im\,\Omega)^{-1}\;{}^t(dZ)\,d\overline{Z})\right)$$

is a Kähler metric on A_Ω invariant under the action (1.2) of $\Gamma^J = Sp(g,\mathbb{Z}) \ltimes H_{\mathbb{Z}}^{(h,g)}$ on (Ω, Z) with Ω fixed. Its Laplacian Δ_Ω of ds^2_Ω is given by

$$\Delta_\Omega = \sigma\left((Im\;\Omega)\,\frac{\partial}{\partial Z}\,{}^t\!\left(\frac{\partial}{\partial \overline{Z}}\right)\right).$$

Proof. Let $\tilde{\gamma} = (\gamma, (\lambda, \mu; \kappa)) \in \Gamma^J$ with $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g,\mathbb{Z})$ and $(\tilde{\Omega}, \tilde{Z}) = \tilde{\gamma}\cdot(\Omega, Z)$ with $\Omega \in \mathbb{H}_g$ fixed. Then according to [4, p. 33],

$$\text{Im}\;\gamma\cdot\Omega = {}^t(C\overline{\Omega}+D)^{-1}\,\text{Im}\;\Omega\,(C\Omega+D)^{-1}$$

and by (1.2),

$$d\tilde{Z} = dZ\,(C\Omega+D)^{-1}.$$

Therefore

$$\begin{aligned} & (\text{Im}\;\tilde{\Omega})^{-1}\,{}^t(d\tilde{Z})\,d\overline{\tilde{Z}} \\ = \;& (C\overline{\Omega}+D)\,(\text{Im}\;\Omega)^{-1}\,{}^t(C\Omega+D)\,{}^t(C\Omega+D)^{-1}\,{}^t(dZ)\,d\overline{Z}\,(C\overline{\Omega}+D)^{-1} \\ = \;& (C\overline{\Omega}+D)\,(\text{Im}\;\Omega)^{-1}\,{}^t(dZ)\,d\overline{Z}\,(C\overline{\Omega}+D)^{-1}. \end{aligned}$$

The metric $ds_{iI_g} = \sigma(dZ\,{}^t(d\overline{Z}))$ at $Z = 0$ is positive definite. Since G^J acts on $\mathbb{H}_{g,h}$ transitively, ds^2_Ω is a Riemannian metric for any $\Omega \in \mathbb{H}_g$. We note that the differential operator Δ_Ω is invariant under the action of Γ^J. In fact, according to (1.2),

$$\frac{\partial}{\partial \tilde{Z}} = (C\Omega+D)\,\frac{\partial}{\partial Z}.$$

Hence if f is a differentiable function on A_Ω, then

$$\begin{aligned} & \mathrm{Im}\,\tilde{\Omega}\,\frac{\partial}{\partial\tilde{Z}}\,{}^t\!\left(\frac{\partial f}{\partial\overline{\tilde{Z}}}\right) \\ = \quad & {}^t(C\overline{\Omega}+D)^{-1}\,(\mathrm{Im}\,\Omega)\,(C\Omega+D)^{-1}(C\Omega+D)\,\frac{\partial}{\partial Z}\,{}^t\!\left((C\overline{\Omega}+D)\frac{\partial f}{\partial\overline{Z}}\right) \\ = \quad & {}^t(C\overline{\Omega}+D)^{-1}\,\mathrm{Im}\,\Omega\,\frac{\partial}{\partial Z}\,{}^t\!\left(\frac{\partial f}{\partial\overline{Z}}\right)\,{}^t(C\overline{\Omega}+D). \end{aligned}$$

Therefore

$$\sigma\left(\mathrm{Im}\,\tilde{\Omega}\,\frac{\partial}{\partial\tilde{Z}}\,{}^t\!\left(\frac{\partial}{\partial\overline{\tilde{Z}}}\right)\right) = \sigma\left(\mathrm{Im}\,\Omega\,\frac{\partial}{\partial Z}\,{}^t\!\left(\frac{\partial f}{\partial\overline{Z}}\right)\right).$$

By the induction on h, we can compute the Laplacian Δ_Ω.

□

We let $L^2(A_\Omega)$ be the space of all functions $f: A_\Omega \longrightarrow \mathbb{C}$ such that

$$||f||_\Omega := \int_{A_\Omega} |f(Z)|^2 dv_\Omega,$$

where dv_Ω is the volume element on A_Ω normalized so that $\int_{A_\Omega} dv_\Omega = 1$. The inner product $(\ ,\ )_\Omega$ on the Hilbert space $L^2(A_\Omega)$ is given by

$$(f,g)_\Omega := \int_{A_\Omega} f(Z)\,\overline{g(Z)}\,dv_\Omega, \quad f,g \in L^2(A_\Omega). \tag{4.1}$$

Theorem 4.3. *The set* $\{\, E_{\Omega;A,B} \mid A,B \in \mathbb{Z}^{(h,g)} \,\}$ *is a complete orthonormal basis for* $L^2(A_\Omega)$*. Moreover we have the following spectral decomposition of* Δ_Ω*:*

$$L^2(A_\Omega) = \oplus_{A,B\in\mathbb{Z}^{(h,g)}} \mathbb{C}\cdot E_{\Omega;A,B}.$$

Proof. Let

$$T = \mathbb{C}^{(h,g)}/(\mathbb{Z}^{(h,g)} \times \mathbb{Z}^{(h,g)}) = (\mathbb{R}^{(h,g)} \times \mathbb{R}^{(h,g)})/(\mathbb{Z}^{(h,g)} \times \mathbb{Z}^{(h,g)})$$

be the torus of real dimension $2hg$. The Hilbert space $L^2(T)$ is isomorphic to the $2hg$ tensor product of $L^2(\mathbb{R}/\mathbb{Z})$, where $\mathbb{R}/\mathbb{Z}$ is the one-dimensional real torus. Since $L^2(\mathbb{R}/\mathbb{Z}) = \oplus_{n\in\mathbb{Z}}\mathbb{C}\cdot e^{2\pi i n x}$, the Hilbert space $L^2(T)$ is

$$L^2(T) = \oplus_{A,B\in\mathbb{Z}^{(h,g)}} \mathbb{C}\cdot E_{A,B}(W),$$

where $W = P + iQ,\ P,Q \in \mathbb{R}^{(h,g)}$ and

$$E_{A,B}(W) := e^{2\pi i\,\sigma({}^tAP + {}^tBQ)}, \quad A,B \in \mathbb{Z}^{(h,g)}.$$

The inner product on $L^2(T)$ is defined by

$$(4.2)\quad (f,g) := \int_0^1 \cdots \int_0^1 f(W)\,\overline{g(W)}\,dp_{11}\cdots dp_{hg}dq_{11}\cdots dq_{hg}, \quad f,g\in L^2(T),$$

where $W = P + iQ \in T,\ P = (p_{kl})$ and $Q = (q_{kl})$. Then we see that the set $\left\{E_{A,B}(W)\mid A,B\in\mathbb{Z}^{(h,g)}\right\}$ is a complete orthonormal basis for $L^2(T)$, and each $E_{A,B}(W)$ is an eigenfunction of the standard Laplacian

$$\Delta_T = \sum_{k=}^{h}\sum_{l=1}^{g}\left(\frac{\partial^2}{\partial p_{kl}^2} + \frac{\partial^2}{\partial q_{kl}^2}\right).$$

We define the mapping $\Phi_\Omega : T \longrightarrow A_\Omega$ by

$$(4.3)\qquad \Phi_\Omega(P+iQ) = (P+QX) + i\,QY, \quad P+i\,Q\in T,\ P,Q\in\mathbb{R}^{(h,g)}.$$

This is well defined. We can see that Φ_Ω is a diffeomorphism and that the inverse Φ_Ω^{-1} of Φ_Ω is given by

$$(4.4)\ \ \Phi_\Omega^{-1}(U+iV) = (U - VY^{-1}X) + i\,VY^{-1}, \quad U+iV\in A_\Omega,\ U,V\in\mathbb{R}^{(h,g)}.$$

Using (4.4), we can show that for $A,B\in\mathbb{Z}^{(h,g)}$, the function $E_{A,B}(W)$ on T is transformed to the function $E_{\Omega;A,B}$ on A_Ω via the diffeomorphism Φ_Ω. Using (4.2) and the diffeomorphism Φ_Ω, we can choose a normalized volume element dv_Ω on A_Ω and then we get the inner product on $L^2(A_\Omega)$ defined by (4.1). This completes the proof. □

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