INVARIANT DIFFERENTIAL OPERATORS ON SIEGEL-JACOBI SPACE

JAE-HYUN YANG

ABSTRACT. For two positive integers m and n, we let $\mathbb{H}_n$ be the Siegel upper half plane of degree n and let $\mathbb{C}^{(m,n)}$ be the set of all $m \times n$ complex matrices. In this article, we study differential operators on the Siegel-Jacobi space $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$ that are invariant under the *natural* action of the Jacobi group $Sp(n,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)}$ on $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$, where $H_{\mathbb{R}}^{(n,m)}$ denotes the Heisenberg group. We give some explicit invariant differential operators. We present important problems which are natural. We give some partial solutions for these natural problems.

1. Introduction

For a given fixed positive integer n, we let

$$\mathbb{H}_n = \{\, \Omega \in \mathbb{C}^{(n,n)} \mid \Omega = {}^t\Omega, \quad \operatorname{Im}\Omega > 0 \,\}$$

be the Siegel upper half plane of degree n and let

$$Sp(n,\mathbb{R}) = \{ M \in \mathbb{R}^{(2n,2n)} \mid {}^tM J_n M = J_n \,\}$$

be the symplectic group of degree n, where $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F for two positive integers k and l, ${}^tM$ denotes the transpose matrix of a matrix M and

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

$Sp(n,\mathbb{R})$ acts on $\mathbb{H}_n$ transitively by

$$M \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1}, \tag{1.1}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R})$ and $\Omega \in \mathbb{H}_n$.

For two positive integers m and n, we consider the Heisenberg group

$$H_{\mathbb{R}}^{(n,m)} = \left\{\, (\lambda,\mu;\kappa) \mid \lambda,\mu \in \mathbb{R}^{(m,n)},\ \kappa \in \mathbb{R}^{(m,m)},\ \kappa + \mu\,{}^t\lambda \text{ symmetric} \,\right\}$$

2000 Mathematics Subject Classification. Primary 13A50, 32Wxx, 15A72.
Keywords and phrases : invariants, invariant differential operators, Siegel-Jacobi space.
This work was supported by Basic Science Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology (41493-01) and partially supported by the Max-Planck-Institut für Mathematik in Bonn.

endowed with the following multiplication law

$$\big(\lambda,\mu;\kappa\big)\circ\big(\lambda',\mu';\kappa'\big)=\big(\lambda+\lambda',\mu+\mu';\kappa+\kappa'+\lambda\,{}^t\mu'-\mu\,{}^t\lambda'\big)$$

with $\big(\lambda,\mu;\kappa\big),\big(\lambda',\mu';\kappa'\big)\in H_{\mathbb R}^{(n,m)}$. We define the semidirect product of $Sp(n,\mathbb R)$ and $H_{\mathbb R}^{(n,m)}$

$$G^J=Sp(n,\mathbb R)\ltimes H_{\mathbb R}^{(n,m)}$$

endowed with the following multiplication law

$$\big(M,(\lambda,\mu;\kappa)\big)\cdot\big(M',(\lambda',\mu';\kappa'\,)\big)=\,\big(MM',(\tilde\lambda+\lambda',\tilde\mu+\mu';\kappa+\kappa'+\tilde\lambda\,{}^t\!\mu'-\tilde\mu\,{}^t\!\lambda'\,)\big)$$

with $M,M'\in Sp(n,\mathbb R),(\lambda,\mu;\kappa),\,(\lambda',\mu';\kappa')\in H_{\mathbb R}^{(n,m)}$ and $(\tilde\lambda,\tilde\mu)=(\lambda,\mu)M'$. Then G^J acts on $\mathbb H_n\times\mathbb C^{(m,n)}$ transitively by

$$\big(M,(\lambda,\mu;\kappa)\big)\cdot(\Omega,Z)=\Big(M\cdot\Omega,(Z+\lambda\Omega+\mu)(C\Omega+D)^{-1}\Big), \tag{1.2}$$

where $M=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in Sp(n,\mathbb R),\ (\lambda,\mu;\kappa)\in H_{\mathbb R}^{(n,m)}$ and $(\Omega,Z)\in\mathbb H_n\times\mathbb C^{(m,n)}$. We note that the Jacobi group G^J is *not* a reductive Lie group and that the homogeneous space $\mathbb H_n\times\mathbb C^{(m,n)}$ is not a symmetric space. We refer to [1, 6, 22, 23, 24, 25, 27, 28, 29, 30, 31] about automorphic forms on G^J and topics related to the content of this paper. From now on, for brevity we write $\mathbb H_{n,m}=\mathbb H_n\times\mathbb C^{(m,n)}$, called the Siegel-Jacobi space of degree n and index m.

The aim of this paper is to study differential operators on $\mathbb H_{n,m}$ which are invariant under the *natural* action (1.2) of G^J. The study of these invariant differential operators on the Siegel-Jacobi space $\mathbb H_{n,m}$ is interesting and important in the aspects of invariant theory, arithmetic and geometry. This article is organized as follows. In Section 2, we review differential operators on $\mathbb H_n$ invariant under the action (1.1) of $Sp(n,\mathbb R)$. We let $\mathbb D(\mathbb H_n)$ denote the algebra of all differential operators on $\mathbb H_n$ that are invariant under the action (1.1). According to the work of Harish-Chandra [7, 8], we see that $\mathbb D(\mathbb H_n)$ is a commutative algebra which is isomorphic to the center of the universal enveloping algebra of the complexification of the Lie algebra of $Sp(n,\mathbb R)$. We briefly describe the work of Maass [14] about constructing explicit algebraically independent generators of $\mathbb D(\mathbb H_n)$ and Shimura's construction [18] of canonically defined algebraically independent generators of $\mathbb D(\mathbb H_n)$. In Section 3, we study differential operators on $\mathbb H_{n,m}$ invariant under the action (1.2) of G^J. For two positive integers m and n, we let

$$T_{n,m}=\,\big\{\,(\omega,z)\,|\,\omega=\,{}^t\omega\in\mathbb C^{(n,n)},\ z\in\mathbb C^{(m,n)}\,\big\}$$

be the complex vector space of dimension $\frac{n(n+1)}{2}+mn$. From the adjoint action of the Jacobi group G^J, we have the *natural action* of the unitary group $U(n)$ on $T_{n,m}$ given by

$$u\cdot(\omega,z)=\,(u\,\omega\,{}^tu,z\,{}^tu),\quad u\in U(n),\ (\omega,z)\in T_{n,m}. \tag{1.3}$$

The action (1.3) of $U(n)$ induces canonically the representation τ of $U(n)$ on the polynomial algebra $\mathrm{Pol}(T_{n,m})$ consisting of complex valued polynomial functions on $T_{n,m}$. Let $\mathrm{Pol}(T_{n,m})^{U(n)}$ denote the subalgebra of $\mathrm{Pol}(T_{n,m})$ consisting of all polynomials on $T_{n,m}$ invariant under the representation τ of $U(n)$, and $\mathbb{D}(\mathbb{H}_{n,m})$ denote the algebra of all differential operators on $\mathbb{H}_{n,m}$ invariant under the action (1.2) of G^J. We see that there is a canonically defined linear bijection of $\mathrm{Pol}(T_{n,m})^{U(n)}$ onto $\mathbb{D}(\mathbb{H}_{n,m})$ which is not multiplicative. We will see that $\mathbb{D}(\mathbb{H}_{n,m})$ is *not* commutative. The main important problem is to find explicit generators of $\mathrm{Pol}(T_{n,m})^{U(n)}$ and explicit generators of $\mathbb{D}(\mathbb{H}_{n,m})$. We propose several natural problems. We want to mention that at this moment it is quite complicated and difficult to find the explicit generators of $\mathbb{D}(\mathbb{H}_{n,m})$ and to express invariant differential operators on $\mathbb{H}_{n,m}$ explicitly. In Section 4, we gives some examples of explicit G^J-invariant differential operators on $\mathbb{H}_{n,m}$ that are obtained by complicated calculations. In Section 5, we deal with the special case $n = m = 1$ in detail. We give complete solutions of the problems that are proposed in Section 3. In Section 6, we deal with the case that $n = 1$ and m is arbitrary. We give some partial solutions for the problems proposed in Section 3. In the final section, using these invariant differential operators on the Siegel-Jacobi space, we discuss a notion of Maass-Jacobi forms.

Acknowledgements: This work was in part done during my stay at the Max-Planck-Institut für Mathematik in Bonn. I am very grateful for the hospitality and financial support. I also thank the National Research Foundation of Korea for its financial support. Finally I would like to give my hearty thanks to Don Zagier, Eberhard Freitag, Rainer Weissauer, Hiroyuki Ochiai and Minoru Itoh for their interests in this work and fruitful discussions.

Notations: We denote by $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the field of rational numbers, the field of real numbers and the field of complex numbers respectively. We denote by $\mathbb{Z}$ and $\mathbb{Z}^+$ the ring of integers and the set of all positive integers respectively. The symbol ":=" means that the expression on the right is the definition of that on the left. For two positive integers k and l, $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F. For a square matrix $A \in F^{(k,k)}$ of degree k, $\mathrm{tr}(A)$ denotes the trace of A. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transpose matrix of M. I_n denotes the identity matrix of degree n. For $A \in F^{(k,l)}$ and $B \in F^{(k,k)}$, we set $B[A] = {}^tABA$. For a complex matrix A, $\overline{A}$ denotes the complex *conjugate* of A. For $A \in \mathbb{C}^{(k,l)}$ and $B \in \mathbb{C}^{(k,k)}$, we use the abbreviation $B\{A\} = {}^t\overline{A}BA$. For a positive integer n, I_n denotes the identity matrix of degree n. For a complex number z, $|z|$ denotes the absolute value of z. For a complex number z, $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$ denote the real part of z and the imaginary part of z respectively.

2. Invariant Differential Operators on the Siegel Space

For a coordinate $\Omega=(\omega_{ij})\in\mathbb{H}_n$, we write $\Omega=X+i\,Y$ with $X=(x_{ij}),\ Y=(y_{ij})$ real. We put $d\Omega=\big(d\omega_{ij}\big)$ and $d\overline{\Omega}=\big(d\overline{\omega}_{ij}\big)$. We also put

$$\frac{\partial}{\partial\Omega}=\left(\,\frac{1+\delta_{ij}}{2}\,\frac{\partial}{\partial\omega_{ij}}\,\right)\qquad\text{and}\qquad \frac{\partial}{\partial\overline{\Omega}}=\left(\,\frac{1+\delta_{ij}}{2}\,\frac{\partial}{\partial\overline{\omega}_{ij}}\,\right).$$

Then for a positive real number A,

$$ds^2_{n;A}=A\,\mathrm{tr}\Big(Y^{-1}d\Omega\,Y^{-1}d\overline{\Omega}\Big) \tag{2.1}$$

is a $Sp(n,\mathbb{R})$-invariant Kähler metric on $\mathbb{H}_n$ (cf. [19, 20]), where $\mathrm{tr}(M)$ denotes the trace of a square matrix M. H. Maass [13] proved that the Laplacian of $ds^2_{n;A}$ is given by

$$\Delta_{n;A}=\,\frac{4}{A}\,\mathrm{tr}\left(Y\,{}^t\!\left(Y\frac{\partial}{\partial\overline{\Omega}}\right)\frac{\partial}{\partial\Omega}\right). \tag{2.2}$$

And

$$dv_n(\Omega)=(\det Y)^{-(n+1)}\prod_{1\le i\le j\le n}dx_{ij}\prod_{1\le i\le j\le n}dy_{ij}$$

is a $Sp(n,\mathbb{R})$-invariant volume element on $\mathbb{H}_n$ (cf. [20, p. 130]).

For brevity, we write $G=Sp(n,\mathbb{R})$. The isotropy subgroup K at iI_n for the action (1.1) is a maximal compact subgroup given by

$$K=\left\{\begin{pmatrix}A & -B\\ B & A\end{pmatrix}\,\Big|\;A\,{}^tA+B\,{}^tB=I_n,\ A\,{}^tB=B\,{}^tA,\ A,B\in\mathbb{R}^{(n,n)}\right\}.$$

Let $\mathfrak{k}$ be the Lie algebra of K. Then the Lie algebra $\mathfrak{g}$ of G has a Cartan decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$, where

$$\mathfrak{g}=\left\{\begin{pmatrix}X_1 & X_2\\ X_3 & -\,{}^tX_1\end{pmatrix}\,\Big|\;X_1,X_2,X_3\in\mathbb{R}^{(n,n)},\ X_2=\,{}^tX_2,\ X_3=\,{}^tX_3\right\},$$

$$\mathfrak{k}=\left\{\begin{pmatrix}X & -Y\\ Y & X\end{pmatrix}\in\mathbb{R}^{(2n,2n)}\,\Big|\;{}^tX+X=0,\ Y=\,{}^tY\right\},$$

$$\mathfrak{p}=\left\{\begin{pmatrix}X & Y\\ Y & -X\end{pmatrix}\,\Big|\;X=\,{}^tX,\ Y=\,{}^tY,\ X,Y\in\mathbb{R}^{(n,n)}\right\}.$$

The subspace $\mathfrak{p}$ of $\mathfrak{g}$ may be regarded as the tangent space of $\mathbb{H}_n$ at iI_n. The adjoint representation of G on $\mathfrak{g}$ induces the action of K on $\mathfrak{p}$ given by

$$k\cdot Z=\,kZ\,{}^tk,\quad k\in K,\ Z\in\mathfrak{p}. \tag{2.3}$$

Let T_n be the vector space of $n\times n$ symmetric complex matrices. We let $\Psi:\mathfrak{p}\longrightarrow T_n$ be the map defined by

$$\Psi\left(\begin{pmatrix}X & Y\\ Y & -X\end{pmatrix}\right)=\,X\,+\,i\,Y,\quad \begin{pmatrix}X & Y\\ Y & -X\end{pmatrix}\in\mathfrak{p}. \tag{2.4}$$

We let $\delta : K \longrightarrow U(n)$ be the isomorphism defined by

$$\delta\left(\begin{pmatrix} A & -B \\ B & A \end{pmatrix}\right) = A + i\,B, \quad \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in K, \tag{2.5}$$

where $U(n)$ denotes the unitary group of degree n. We identify $\mathfrak{p}$ (resp. K) with T_n (resp. $U(n)$) through the map Ψ (resp. δ). We consider the action of $U(n)$ on T_n defined by

$$h \cdot \omega = h\omega\, {}^t h, \quad h \in U(n),\ \omega \in T_n. \tag{2.6}$$

Then the adjoint action (2.3) of K on $\mathfrak{p}$ is compatible with the action (2.6) of $U(n)$ on T_n through the map Ψ. Precisely for any $k \in K$ and $Z \in \mathfrak{p}$, we get

$$\Psi(k\,Z\,{}^t k) = \delta(k)\,\Psi(Z)\,{}^t\delta(k). \tag{2.7}$$

The action (2.6) induces the action of $U(n)$ on the polynomial algebra $\mathrm{Pol}(T_n)$ and the symmetric algebra $S(T_n)$ respectively. We denote by $\mathrm{Pol}(T_n)^{U(n)}$ $\Big($resp. $S(T_n)^{U(n)}\Big)$ the subalgebra of $\mathrm{Pol}(T_n)$ $\Big($resp. $S(T_n)\Big)$ consisting of $U(n)$-invariants. The following inner product $(\ ,\ )$ on T_n defined by

$$(Z, W) = \mathrm{tr}\big(Z\,\overline{W}\,\big), \quad Z, W \in T_n$$

gives an isomorphism as vector spaces

$$T_n \cong T_n^*, \quad Z \mapsto f_Z, \quad Z \in T_n, \tag{2.8}$$

where T_n^* denotes the dual space of T_n and f_Z is the linear functional on T_n defined by

$$f_Z(W) = (W, Z), \quad W \in T_n.$$

It is known that there is a canonical linear bijection of $S(T_n)^{U(n)}$ onto the algebra $\mathbb{D}(\mathbb{H}_n)$ of differential operators on $\mathbb{H}_n$ invariant under the action (1.1) of G. Identifying T_n with T_n^* by the above isomorphism (2.8), we get a canonical linear bijection

$$\Theta_n : \mathrm{Pol}(T_n)^{U(n)} \longrightarrow \mathbb{D}(\mathbb{H}_n) \tag{2.9}$$

of $\mathrm{Pol}(T_n)^{U(n)}$ onto $\mathbb{D}(\mathbb{H}_n)$. The map Θ_n is described explicitly as follows. Similarly the action (2.3) induces the action of K on the polynomial algebra $\mathrm{Pol}(\mathfrak{p})$ and the symmetric algebra $S(\mathfrak{p})$ respectively. Through the map Ψ, the subalgebra $\mathrm{Pol}(\mathfrak{p})^K$ of $\mathrm{Pol}(\mathfrak{p})$ consisting of K-invariants is isomorphic to $\mathrm{Pol}(T_n)^{U(n)}$. We put $N = n(n+1)$. Let $\{\xi_\alpha \mid 1 \le \alpha \le N\}$ be a basis of a real vector space $\mathfrak{p}$. If $P \in \mathrm{Pol}(\mathfrak{p})^K$, then

$$\Big(\Theta_n(P)f\Big)(gK) = \left[P\left(\frac{\partial}{\partial t_\alpha}\right) f\left(g\exp\left(\sum_{\alpha=1}^N t_\alpha \xi_\alpha\right)K\right)\right]_{(t_\alpha)=0}, \tag{2.10}$$

where $f \in C^\infty(\mathbb{H}_n)$. We refer to [9, 10] for more detail. In general, it is hard to express $\Phi(P)$ explicitly for a polynomial $P \in \mathrm{Pol}(\mathfrak{p})^K$.

According to the work of Harish-Chandra [7, 8], the algebra $\mathbb{D}(\mathbb{H}_n)$ is generated by n algebraically independent generators and is isomorphic to the commutative ring

$\mathbb{C}[x_1,\cdots,x_n]$ with n indeterminates. We note that n is the real rank of G. Let $\mathfrak{g}_{\mathbb{C}}$ be the complexification of $\mathfrak{g}$. It is known that $\mathbb{D}(\mathbb{H}_n)$ is isomorphic to the center of the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$.

Using a classical invariant theory (cf. [11, 21], we can show that $\mathrm{Pol}(T_n)^{U(n)}$ is generated by the following algebraically independent polynomials

$$q_j(\omega) = \mathrm{tr}\Big(\big(\omega\overline{\omega}\big)^j \Big), \quad \omega \in T_n, \quad j = 1, 2, \cdots, n. \tag{2.11}$$

For each j with $1 \leq j \leq n$, the image $\Theta_n(q_j)$ of q_j is an invariant differential operator on $\mathbb{H}_n$ of degree $2j$. The algebra $\mathbb{D}(\mathbb{H}_n)$ is generated by n algebraically independent generators $\Theta_n(q_1), \Theta_n(q_2), \cdots, \Theta_n(q_n)$. In particular,

$$\Theta_n(q_1) = c_1\, \mathrm{tr}\left(Y\, {}^t\!\left(Y \frac{\partial}{\partial\overline{\Omega}}\right) \frac{\partial}{\partial\Omega}\right) \quad \text{for some constant } c_1. \tag{2.12}$$

We observe that if we take $\omega = x + i\,y \in T_n$ with real x, y, then $q_1(\omega) = q_1(x,y) = \mathrm{tr}\big(x^2 + y^2\big)$ and

$$q_2(\omega) = q_2(x,y) = \mathrm{tr}\Big(\big(x^2+y^2\big)^2 + 2\,x\big(xy - yx\big)y \Big).$$

It is a natural question to express the images $\Theta_n(q_j)$ explicitly for $j = 2, 3, \cdots, n$. We hope that the images $\Theta_n(q_j)$ for $j = 2, 3, \cdots, n$ are expressed in the form of the *trace* as $\Phi(q_1)$.

H. Maass [14] found algebraically independent generators $H_1, H_2, \cdots, H_n$ of $\mathbb{D}(\mathbb{H}_n)$. We will describe $H_1, H_2, \cdots, H_n$ explicitly. For $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R})$ and $\Omega = X + iY \in \mathbb{H}_n$ with real X, Y, we set

$$\Omega_* = M\cdot\Omega = X_* + iY_* \quad \text{with } X_*, Y_* \text{ real}.$$

We set

$$\begin{aligned} K &= \big(\Omega - \overline{\Omega}\,\big)\frac{\partial}{\partial\Omega} = 2\,i\,Y\frac{\partial}{\partial\Omega}, \\ \Lambda &= \big(\Omega - \overline{\Omega}\,\big)\frac{\partial}{\partial\overline{\Omega}} = 2\,i\,Y\frac{\partial}{\partial\overline{\Omega}}, \\ K_* &= \big(\Omega_* - \overline{\Omega}_*\,\big)\frac{\partial}{\partial\Omega_*} = 2\,i\,Y_*\frac{\partial}{\partial\Omega_*}, \\ \Lambda_* &= \big(\Omega_* - \overline{\Omega}_*\,\big)\frac{\partial}{\partial\overline{\Omega}_*} = 2\,i\,Y_*\frac{\partial}{\partial\overline{\Omega}_*}. \end{aligned}$$

Then it is easily seen that

$$K_* = {}^t(C\overline{\Omega} + D)^{-1}\, {}^t\big\{(C\Omega + D)\, {}^tK\big\}, \tag{2.13}$$

$$\Lambda_* = {}^t(C\Omega + D)^{-1}\, {}^t\big\{(C\overline{\Omega} + D)\, {}^t\Lambda\big\} \tag{2.14}$$

and

$$ {}^t\big\{(C\overline{\Omega}+D)\,{}^t\Lambda\big\} = \Lambda\,{}^t(C\overline{\Omega}+D) - \frac{n+1}{2}\,\big(\Omega-\overline{\Omega}\,\big)\,{}^tC. \tag{2.15} $$

Using Formulas (2.13), (2.14) and (2.15), we can show that

$$ \Lambda_* K_* + \frac{n+1}{2}K_* = {}^t(C\Omega+D)^{-1}\,{}^t\Big\{(C\Omega+D)\,{}^t\Big(\Lambda K + \frac{n+1}{2}K\Big)\Big\}. \tag{2.16} $$

Therefore we get

$$ \mathrm{tr}\Big(\Lambda_* K_* + \frac{n+1}{2}K_*\Big) = \mathrm{tr}\Big(\Lambda K + \frac{n+1}{2}K\Big). \tag{2.17} $$

We set

$$ A^{(1)} = \Lambda K + \frac{n+1}{2}K. \tag{2.18} $$

We define $A^{(j)}\,(j=2,3,\cdots,n)$ recursively by

$$ \begin{aligned} A^{(j)} \;&=\; A^{(1)}A^{(j-1)} - \frac{n+1}{2}\,\Lambda\,A^{(j-1)} + \frac{1}{2}\,\Lambda\,\mathrm{tr}\big(A^{(j-1)}\big) \\ &\quad + \frac{1}{2}\,\big(\Omega-\overline{\Omega}\,\big)\,{}^t\Big\{\big(\Omega-\overline{\Omega}\,\big)^{-1}\,{}^t\big({}^t\Lambda\,{}^tA^{(j-1)}\big)\Big\}. \end{aligned} \tag{2.19} $$

We set

$$ H_j = \mathrm{tr}\big(A^{(j)}\big), \quad j=1,2,\cdots,n. \tag{2.20} $$

As mentioned before, Maass proved that $H_1, H_2, \cdots, H_n$ are algebraically independent generators of $\mathbb{D}(\mathbb{H}_n)$.

In fact, we see that

$$ -H_1 = \Delta_{n;1} = 4\,\mathrm{tr}\Big(Y\,{}^t\Big(Y\frac{\partial}{\partial\overline{\Omega}}\Big)\frac{\partial}{\partial\Omega}\Big). \tag{2.21} $$

is the Laplacian for the invariant metric $ds^2_{n;1}$ on $\mathbb{H}_n$.

Conjecture. For $j=2,3,\cdots,n,\ \Theta_n(q_j) = c_j\,H_j$ for a suitable constant c_j.

Example 2.1. We consider the case $n=1$. The algebra $\mathrm{Pol}(T_1)^{U(1)}$ is generated by the polynomial

$$ q(\omega) = \omega\,\overline{\omega}, \quad \omega = x + iy \in \mathbb{C} \text{ with } x, y \text{ real}. $$

Using Formula (2.10), we get

$$ \Theta_1(q) = 4\,y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right). $$

Therefore $\mathbb{D}(\mathbb{H}_1) = \mathbb{C}\big[\Theta_1(q)\big] = \mathbb{C}[H_1]$.

Example 2.2. We consider the case $n=2$. The algebra $\mathrm{Pol}(T_2)^{U(2)}$ is generated by the polynomial

$$q_1(\omega)=\mathrm{tr}\big(\omega\,\overline{\omega}\,\big),\quad q_2(\omega)=\mathrm{tr}\Big(\big(\omega\,\overline{\omega}\big)^2\Big),\quad \omega\in T_2.$$

Using Formula (2.10), we may express $\Theta_2(q_1)$ and $\Theta_2(q_2)$ explicitly. $\Theta_2(q_1)$ is expressed by Formula (2.12). The computation of $\Theta_2(q_2)$ might be quite tedious. We leave the detail to the reader. In this case, $\Theta_2(q_2)$ was essentially computed in [4], Proposition 6. Therefore

$$\mathbb{D}(\mathbb{H}_2)=\mathbb{C}\big[\Theta_2(q_1),\Theta_2(q_2)\big]=\mathbb{C}[H_1,H_2].$$

In fact, the center of the universal enveloping algebra $\mathscr{U}(\mathfrak{g}_{\mathbb{C}})$ was computed in [4].

G. Shimura [18] found canonically defined algebraically independent generators of $\mathbb{D}(\mathbb{H}_n)$. We will describe his way of constructing those generators roughly. Let $K_{\mathbb{C}},\ \mathfrak{g}_{\mathbb{C}},\ \mathfrak{k}_{\mathbb{C}},\mathfrak{p}_{\mathbb{C}},\cdots$ denote the complexication of $K,\ \mathfrak{g},\ \mathfrak{k},\ \mathfrak{p},\cdots$ respectively. Then we have the Cartan decomposition

$$\mathfrak{g}_{\mathbb{C}}=\mathfrak{k}_{\mathbb{C}}+\mathfrak{p}_{\mathbb{C}},\quad \mathfrak{p}_{\mathbb{C}}=\mathfrak{p}_{\mathbb{C}}^{+}+\mathfrak{p}_{\mathbb{C}}^{-}$$

with the properties

$$[\mathfrak{k}_{\mathbb{C}},\mathfrak{p}_{\mathbb{C}}^{\pm}]\subset\mathfrak{p}_{\mathbb{C}}^{\pm},\quad [\mathfrak{p}_{\mathbb{C}}^{+},\mathfrak{p}_{\mathbb{C}}^{+}]=[\mathfrak{p}_{\mathbb{C}}^{-},\mathfrak{p}_{\mathbb{C}}^{-}]=\{0\},\quad [\mathfrak{p}_{\mathbb{C}}^{+},\mathfrak{p}_{\mathbb{C}}^{-}]=\mathfrak{k}_{\mathbb{C}},$$

where

$$\mathfrak{g}_{\mathbb{C}}=\left\{\begin{pmatrix}X_1 & X_2\\ X_3 & -\,{}^tX_1\end{pmatrix}\ \Big|\ X_1,X_2,X_3\in\mathbb{C}^{(n,n)},\ X_2={}^tX_2,\ X_3={}^tX_3\right\},$$

$$\mathfrak{k}_{\mathbb{C}}=\left\{\begin{pmatrix}A & -B\\ B & A\end{pmatrix}\in\mathbb{C}^{(2n,2n)}\ \Big|\ {}^tA+A=0,\ B={}^tB\right\},$$

$$\begin{aligned}\mathfrak{p}_{\mathbb{C}}&=\left\{\begin{pmatrix}X & Y\\ Y & -X\end{pmatrix}\in\mathbb{C}^{(2n,2n)}\ \Big|\ X={}^tX,\ Y={}^tY\right\},\\ \mathfrak{p}_{\mathbb{C}}^{+}&=\left\{\begin{pmatrix}Z & iZ\\ iZ & -Z\end{pmatrix}\in\mathbb{C}^{(2n,2n)}\ \Big|\ Z={}^tZ\in\mathbb{C}^{(n,n)}\right\},\\ \mathfrak{p}_{\mathbb{C}}^{-}&=\left\{\begin{pmatrix}Z & -iZ\\ -iZ & -Z\end{pmatrix}\in\mathbb{C}^{(2n,2n)}\ \Big|\ Z={}^tZ\in\mathbb{C}^{(n,n)}\right\}.\end{aligned}$$

For a complex vector space W and a nonnegative integer r, we denote by $\mathrm{Pol}_r(W)$ the vector space of complex-valued homogeneous polynomial functions on W of degree r. We put

$$\mathrm{Pol}^r(W):=\sum_{s=0}^{r}\mathrm{Pol}_s(W).$$

$\mathrm{Ml}_r(W)$ denotes the vector space of all $\mathbb{C}$-multilinear maps of $W\times\cdots\times W$ (r copies) into $\mathbb{C}$. An element Q of $\mathrm{Ml}_r(W)$ is called *symmetric* if

$$Q(x_1,\cdots,x_r)=Q(x_{\pi(1)},\cdots,x_{\pi(r)})$$

for each permutation π of $\{1, 2, \cdots, r\}$. Given $P \in \mathrm{Pol}_r(W)$, there is a unique element symmetric element P_* of $\mathrm{Ml}_r(W)$ such that

$$P(x) = P_*(x, \cdots, x) \qquad \text{for all } x \in W. \tag{2.22}$$

Moreover the map $P \mapsto P_*$ is a $\mathbb{C}$-linear bijection of $\mathrm{Pol}_r(W)$ onto the set of all symmetric elements of $\mathrm{Ml}_r(W)$. We let $S_r(W)$ denote the subspace consisting of all homogeneous elements of degree r in the symmetric algebra $S(W)$. We note that $\mathrm{Pol}_r(W)$ and $S_r(W)$ are dual to each other with respect to the pairing

$$\langle \alpha, x_1 \cdots x_r \rangle = \alpha_*(x_1, \cdots, x_r) \qquad (x_i \in W,\ \alpha \in \mathrm{Pol}_r(W)). \tag{2.23}$$

Let $\mathfrak{p}_{\mathbb{C}}^*$ be the dual space of $\mathfrak{p}_{\mathbb{C}}$, that is, $\mathfrak{p}_{\mathbb{C}}^* = \mathrm{Pol}_1(\mathfrak{p}_{\mathbb{C}})$. Let $\{X_1, \cdots, X_N\}$ be a basis of $\mathfrak{p}_{\mathbb{C}}$ and $\{Y_1, \cdots, Y_N\}$ be the basis of $\mathfrak{p}_{\mathbb{C}}^*$ dual to $\{X_\nu\}$, where $N = n(n+1)$. We note that $\mathrm{Pol}_r(\mathfrak{p}_{\mathbb{C}})$ and $\mathrm{Pol}_r(\mathfrak{p}_{\mathbb{C}}^*)$ are dual to each other with respect to the pairing

$$\langle \alpha, \beta \rangle = \sum \alpha_*(X_{i_1}, \cdots, X_{i_r})\, \beta_*(Y_{i_1}, \cdots, Y_{i_r}), \tag{2.24}$$

where $\alpha \in \mathrm{Pol}_r(\mathfrak{p}_{\mathbb{C}}),\ \beta \in \mathrm{Pol}_r(\mathfrak{p}_{\mathbb{C}}^*)$ and $(i_1, \cdots, i_r)$ runs over $\{1, \cdots, N\}^r$. Let $\mathscr{U}(\mathfrak{g}_{\mathbb{C}})$ be the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$ and $\mathscr{U}^p(\mathfrak{g}_{\mathbb{C}})$ its subspace spanned by the elements of the form $V_1 \cdots V_s$ with $V_i \in \mathfrak{g}_{\mathbb{C}}$ and $s \leq p$. We recall that there is a $\mathbb{C}$-linear bijection ψ of the symmetric algebra $S(\mathfrak{g}_{\mathbb{C}})$ of $\mathfrak{g}_{\mathbb{C}}$ onto $\mathscr{U}(\mathfrak{g}_{\mathbb{C}})$ which is characterized by the property that $\psi(X^r) = X^r$ for all $X \in \mathfrak{g}_{\mathbb{C}}$. For each $\alpha \in \mathrm{Pol}_r(\mathfrak{p}_{\mathbb{C}}^*)$ we define an element $\omega(\alpha)$ of $\mathscr{U}(\mathfrak{g}_{\mathbb{C}})$ by

$$\omega(\alpha) := \sum \alpha_*(Y_{i_1}, \cdots, Y_{i_r})\, X_{i_1} \cdots X_{i_r}, \tag{2.25}$$

where $(i_1, \cdots, i_r)$ runs over $\{1, \cdots, N\}^r$. If $Y \in \mathfrak{p}_{\mathbb{C}}$, then Y^r as an element of $\mathrm{Pol}_r(\mathfrak{p}_{\mathbb{C}}^*)$ is defined by

$$Y^r(u) = Y(u)^r \qquad \text{for all } u \in \mathfrak{p}_{\mathbb{C}}^*.$$

Hence $(Y^r)_*(u_1, \cdots, u_r) = Y(u_1) \cdots Y(u_r)$. According to (2.25), we see that if $\alpha(\sum t_i Y_i) = P(t_1, \cdots, t_N)$ for $t_i \in \mathbb{C}$ with a polynomial P, then

$$\omega(\alpha) = \psi(P(X_1, \cdots, X_N)). \tag{2.26}$$

Thus ω is a $\mathbb{C}$-linear injection of $\mathrm{Pol}(\mathfrak{p}_{\mathbb{C}}^*)$ into $\mathscr{U}(\mathfrak{g}_{\mathbb{C}})$ independent of the choice of a basis. We observe that $\omega\big(\mathrm{Pol}_r(\mathfrak{p}_{\mathbb{C}}^*)\big) = \psi(S_r(\mathfrak{p}_{\mathbb{C}}))$. It is a well-known fact that if $\alpha_1, \cdots, \alpha_m \in \mathrm{Pol}_r(\mathfrak{p}_{\mathbb{C}}^*)$, then

$$\omega(\alpha_1 \cdots \alpha_m) - \omega(\alpha_m) \cdots \omega(\alpha_1) \in \mathscr{U}^{r-1}(\mathfrak{g}_{\mathbb{C}}). \tag{2.27}$$

We have a canonical pairing

$$\langle\ ,\ \rangle : \mathrm{Pol}_r(\mathfrak{p}_{\mathbb{C}}^+) \times \mathrm{Pol}_r(\mathfrak{p}_{\mathbb{C}}^-) \longrightarrow \mathbb{C} \tag{2.28}$$

defined by

$$\langle f, g \rangle = \sum f_*(\widetilde{X}_{i_1}, \cdots, \widetilde{X}_{i_r}) g_*(\widetilde{Y}_{i_1}, \cdots, \widetilde{Y}_{i_r}), \tag{2.29}$$

where f_* (resp. g_*) are the unique symmetric elements of $\mathrm{Ml}_r(\mathfrak{p}_{\mathbb{C}}^+)$ (resp. $\mathrm{Ml}_r(\mathfrak{p}_{\mathbb{C}}^-)$), and $\{\widetilde{X}_1, \cdots, \widetilde{X}_{\widetilde{N}}\}$ and $\{\widetilde{Y}_1, \cdots, \widetilde{Y}_{\widetilde{N}}\}$ are dual bases of $\mathfrak{p}_{\mathbb{C}}^+$ and $\mathfrak{p}_{\mathbb{C}}^-$ with respect to the Killing form $B(X,Y) = 2(n+1)\,\mathrm{tr}(XY)$, $\widetilde{N} = \frac{n(n+1)}{2}$, and $(i_1,\cdots,i_r)$ runs over $\big\{1,\cdots,\widetilde{N}\big\}^r$.

The adjoint representation of $K_{\mathbb{C}}$ on $\mathfrak{p}^{\pm}_{\mathbb{C}}$ induces the representation of $K_{\mathbb{C}}$ on $\mathrm{Pol}_r(\mathfrak{p}^{\pm}_{\mathbb{C}})$. Given a $K_{\mathbb{C}}$-irreducible subspace Z of $\mathrm{Pol}_r(\mathfrak{p}^{+}_{\mathbb{C}})$, we can find a unique $K_{\mathbb{C}}$-irreducible subspace W of $\mathrm{Pol}_r(\mathfrak{p}^{-}_{\mathbb{C}})$ such that $\mathrm{Pol}_r(\mathfrak{p}^{-}_{\mathbb{C}})$ is the direct sum of W and the annihilator of Z. Then Z and W are dual with respect to the pairing (2.28). Take bases $\{\zeta_1,\cdots,\zeta_\kappa\}$ of Z and $\{\xi_1,\cdots,\xi_\kappa\}$ of W that are dual to each other. We set

$$f_Z(x,y) = \sum_{\nu=1}^{\kappa} \zeta_\nu(x)\,\xi_\nu(y) \qquad (x\in\mathfrak{p}^{+}_{\mathbb{C}},\ y\in\mathfrak{p}^{-}_{\mathbb{C}}). \tag{2.30}$$

It is easily seen that f_Z belongs to $\mathrm{Pol}_{2r}(\mathfrak{p}_{\mathbb{C}})^K$ and is independent of the choice of dual bases $\{\zeta_\nu\}$ and $\{\xi_\nu\}$. Shimura [18] proved that there exists a canonically defined set $\{Z_1,\cdots,Z_n\}$ with a $K_{\mathbb{C}}$-irreducible subspace Z_r of $\mathrm{Pol}_r(\mathfrak{p}^{+}_{\mathbb{C}})$ $(1\le r\le n)$ such that $f_{Z_1},\cdots,f_{Z_n}$ are algebraically independent generators of $\mathrm{Pol}(\mathfrak{p}_{\mathbb{C}})^K$. We can identify $\mathfrak{p}^{+}_{\mathbb{C}}$ with T_n. We recall that T_n denotes the vector space of $n\times n$ symmetric complex matrices. We can take Z_r as the subspace of $\mathrm{Pol}_r(T_n)$ spanned by the functions $f_{a;r}(Z) = \det_r({}^t a Z a)$ for all $a\in GL(n,\mathbb{C})$, where $\det_r(x)$ denotes the determinant of the upper left $r\times r$ submatrix of x. For every $f\in \mathrm{Pol}(\mathfrak{p}_{\mathbb{C}})^K$, we let $\Omega(f)$ denote the element of $\mathbb{D}(\mathbb{H}_n)$ represented by $\omega(f)$. Then $\mathbb{D}(\mathbb{H}_n)$ is the polynomial ring $\mathbb{C}[\omega(f_{Z_1}),\cdots,\omega(f_{Z_n})]$ generated by n algebraically independent elements $\omega(f_{Z_1}),\cdots,\omega(f_{Z_n})$.

3. Invariant Differential Operators on Siegel-Jacobi Space

The stabilizer K^J of G^J at $(iI_n,0)$ is given by

$$K^J = \Big\{ \big(k,(0,0;\kappa)\big)\ \big|\ k\in K,\ \kappa = {}^t\kappa\in\mathbb{R}^{(m,m)} \Big\}.$$

Therefore $\mathbb{H}_{n,m}\cong G^J/K^J$ is a homogeneous space of *non-reductive type*. The Lie algebra $\mathfrak{g}^J$ of G^J has a decomposition

$$\mathfrak{g}^J = \mathfrak{k}^J + \mathfrak{p}^J,$$

where

$$\mathfrak{g}^J = \Big\{ \big(Z,(P,Q,R)\big)\ \big|\ Z\in\mathfrak{g},\ P,Q\in\mathbb{R}^{(m,n)},\ R={}^tR\in\mathbb{R}^{(m,m)} \Big\},$$

$$\mathfrak{k}^J = \Big\{ \big(X,(0,0,R)\big)\ \big|\ X\in\mathfrak{k},\ R={}^tR\in\mathbb{R}^{(m,m)} \Big\},$$

$$\mathfrak{p}^J = \Big\{ \big(Y,(P,Q,0)\big)\ \big|\ Y\in\mathfrak{p},\ P,Q\in\mathbb{R}^{(m,n)} \Big\}.$$

Thus the tangent space of the homogeneous space $\mathbb{H}_{n,m}$ at $(iI_n,0)$ is identified with $\mathfrak{p}^J$.

If $\alpha = \left(\begin{pmatrix} X_1 & Y_1 \\ Z_1 & -{}^tX_1 \end{pmatrix}, (P_1, Q_1, R_1) \right)$ and $\beta = \left(\begin{pmatrix} X_2 & Y_2 \\ Z_2 & -{}^tX_2 \end{pmatrix}, (P_2, Q_2, R_2) \right)$ are elements of $\mathfrak{g}^J$, then the Lie bracket $[\alpha, \beta]$ of α and β is given by

$$[\alpha, \beta] = \left(\begin{pmatrix} X^* & Y^* \\ Z^* & -{}^tX^* \end{pmatrix}, (P^*, Q^*, R^*) \right), \tag{3.1}$$

where

$$\begin{aligned} X^* &= X_1X_2 - X_2X_1 + Y_1Z_2 - Y_2Z_1, \\ Y^* &= X_1Y_2 - X_2Y_1 + Y_2\,{}^tX_1 - Y_1\,{}^tX_2, \\ Z^* &= Z_1X_2 - Z_2X_1 + {}^tX_2Z_1 - {}^tX_1Z_2, \\ P^* &= P_1X_2 - P_2X_1 + Q_1Z_2 - Q_2Z_1, \\ Q^* &= P_1Y_2 - P_2Y_1 + Q_2\,{}^tX_1 - Q_1\,{}^tX_2, \\ R^* &= P_1\,{}^tQ_2 - P_2\,{}^tQ_1 + Q_2\,{}^tP_1 - Q_1\,{}^tP_2 \end{aligned}$$

Lemma 3.1.

$$[\mathfrak{k}^J, \mathfrak{k}^J] \subset \mathfrak{k}^J, \quad [\mathfrak{k}^J, \mathfrak{p}^J] \subset \mathfrak{p}^J.$$

Proof. The proof follows immediately from Formula (3.1). □

Lemma 3.2. *Let*

$$k^J = \left(\begin{pmatrix} A & -B \\ B & A \end{pmatrix}, (0, 0, \kappa) \right) \in K^J$$

with $\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in K,\ \kappa = {}^t\kappa \in \mathbb{R}^{(m,m)}$ *and*

$$\alpha = \left(\begin{pmatrix} X & Y \\ Y & -X \end{pmatrix}, (P, Q, 0) \right) \in \mathfrak{p}^J$$

with $X = {}^tX,\ Y = {}^tY \in \mathbb{R}^{(n,n)},\ P, Q \in \mathbb{R}^{(m,n)}$. *Then the adjoint action of* K^J *on* $\mathfrak{p}^J$ *is given by*

$$Ad(k^J)\alpha = \left(\begin{pmatrix} X_* & Y_* \\ Y_* & -X_* \end{pmatrix}, (P_*, Q_*, 0) \right), \tag{3.2}$$

where

$$X_* = AX\,{}^tA - \big(BX\,{}^tB + BY\,{}^tA + AY\,{}^tB\big), \tag{3.3}$$

$$Y_* = \big(AX\,{}^tB + AY\,{}^tA + BX\,{}^tA\big) - BY\,{}^tB, \tag{3.4}$$

$$P_* = P\,{}^tA - Q\,{}^tB, \tag{3.5}$$

$$Q_* = P\,{}^tB + Q\,{}^tA. \tag{3.6}$$

Proof. We leave the proof to the reader. □

We recall that T_n denotes the vector space of all $n\times n$ symmetric complex matrices. For brevity, we put $T_{n,m}:=T_n\times\mathbb{C}^{(m,n)}$. We define the real linear map $\Phi:\mathfrak{p}^J\longrightarrow T_{n,m}$ by

$$\Phi\left(\begin{pmatrix} X & Y\\ Y & -X\end{pmatrix},(P,Q,0)\right)=\big(X+i\,Y,\;P+i\,Q\big), \tag{3.7}$$

where $\begin{pmatrix} X & Y\\ Y & -X\end{pmatrix}\in\mathfrak{p}$ and $P,Q\in\mathbb{R}^{(m,n)}$.

Let $S(m,\mathbb{R})$ denote the additive group consisting of all $m\times m$ real symmetric matrices. Now we define the isomorphism $\theta:K^J\longrightarrow U(n)\times S(m,\mathbb{R})$ by

$$\theta(h,(0,0,\kappa))=(\delta(h),\kappa),\quad h\in K,\ \kappa\in S(m,\mathbb{R}), \tag{3.8}$$

where $\delta:K\longrightarrow U(n)$ is the map defined by (2.5). Identifying $\mathbb{R}^{(m,n)}\times\mathbb{R}^{(m,n)}$ with $\mathbb{C}^{(m,n)}$, we can identify $\mathfrak{p}^J$ with $T_n\times\mathbb{C}^{(m,n)}$.

Theorem 3.1. *The adjoint representation of K^J on $\mathfrak{p}^J$ is compatible with the natural action of $U(n)\times S(m,\mathbb{R})$ on $T_{n,m}$ defined by*

$$(h,\kappa)\cdot(\omega,z):=(h\,\omega\,{}^th,\;z\,{}^th),\qquad h\in U(n),\ \kappa\in S(m,\mathbb{R}),\ (\omega,z)\in T_{n,m} \tag{3.9}$$

through the maps Φ and θ. Precisely, if $k^J\in K^J$ and $\alpha\in\mathfrak{p}^J$, then we have the following equality

$$\Phi\big(Ad(k^J)\alpha\big)=\theta(k^J)\cdot\Phi(\alpha). \tag{3.10}$$

Here we regard the complex vector space $T_{n,m}$ as a real vector space.

Proof. Let

$$k^J=\left(\begin{pmatrix} A & -B\\ B & A\end{pmatrix},(0,0,\kappa)\right)\in K^J$$

with $\begin{pmatrix} A & -B\\ B & A\end{pmatrix}\in K,\ \kappa={}^t\kappa\in\mathbb{R}^{(m,m)}$ and

$$\alpha=\left(\begin{pmatrix} X & Y\\ Y & -X\end{pmatrix},(P,Q,0)\right)\in\mathfrak{p}^J$$

with $X={}^tX,\ Y={}^tY\in\mathbb{R}^{(n,n)},\ P,Q\in\mathbb{R}^{(m,n)}$. Then we have

$$\begin{aligned}
\theta(k^J)\cdot\Phi(\alpha) &= \big(A+i\,B,\,\kappa\big)\cdot\big(X+i\,Y,\,P+i\,Q\big)\\
&= \big((A+iB)(X+iY)\,{}^t(A+iB),\;(P+iQ)\,{}^t(A+iB)\big)\\
&= \big(X_*+i\,Y_*,\,P_*+i\,Q_*\big)\\
&= \Phi\left(\begin{pmatrix} X_* & Y_*\\ Y_* & -X_*\end{pmatrix},(P_*,Q_*,0)\right)\\
&= \Phi\big(Ad(k^J)\alpha\big)\qquad (\textit{by Lemma } 3.2),
\end{aligned}$$

where X_*, Y_*, Z_* and Q_* are given by the formulas (3.3), (3.4), (3.5) and (3.6) respectively. □

We now study the algebra $\mathbb{D}(\mathbb{H}_{n,m})$ of all differential operators on $\mathbb{H}_{n,m}$ invariant under the *natural action* (1.2) of G^J. The action (3.9) induces the action of $U(n)$ on the polynomial algebra $\mathrm{Pol}_{n,m} := \mathrm{Pol}\,(T_{n,m})$. We denote by $\mathrm{Pol}_{n,m}^{U(n)}$ the subalgebra of $\mathrm{Pol}_{n,m}$ consisting of all $U(n)$-invariants. Similarly the action (3.2) of K induces the action of K on the polynomial algebra $\mathrm{Pol}(\mathfrak{p}^J)$. We see that through the identification of $\mathfrak{p}^J$ with $T_{n,m}$, the algebra $\mathrm{Pol}(\mathfrak{p}^J)$ is isomorphic to $\mathrm{Pol}_{n,m}$. The following $U(n)$-invariant inner product $(\ ,\ )_*$ of the complex vector space $T_{n,m}$ defined by

$$\big((\omega,z),(\omega',z')\big)_* = \mathrm{tr}\big(\omega\overline{\omega'}\big) + \mathrm{tr}\big(z\,{}^t\overline{z'}\big), \quad (\omega,z),\ (\omega',z') \in T_{n,m}$$

gives a canonical isomorphism

$$T_{n,m} \cong T_{n,m}^*, \quad (\omega,z) \mapsto f_{\omega,z}, \quad (\omega,z) \in T_{n,m},$$

where $f_{\omega,z}$ is the linear functional on $T_{n,m}$ defined by

$$f_{\omega,z}\big((\omega',z'\,)\big) = \big((\omega',z'),(\omega,z)\big)_*, \quad (\omega',z'\,) \in T_{n,m}.$$

According to Helgason ([10], p. 287), one gets a canonical linear bijection of $S(T_{n,m})^{U(n)}$ onto $\mathbb{D}(\mathbb{H}_{n,m})$. Identifying $T_{n,m}$ with $T_{n,m}^*$ by the above isomorphism, one gets a natural linear bijection

$$\Theta_{n,m} :\ \mathrm{Pol}_{n,m}^{U(n)} \longrightarrow \mathbb{D}(\mathbb{H}_{n,m})$$

of $\mathrm{Pol}_{n,m}^{U(n)}$ onto $\mathbb{D}(\mathbb{H}_{n,m})$. The map $\Theta_{n,m}$ is described explicitly as follows. We put $N_\star = n(n+1) + 2mn$. Let $\big\{\eta_\alpha \mid 1 \le \alpha \le N_\star\big\}$ be a basis of $\mathfrak{p}^J$. If $P \in \mathrm{Pol}\big(\mathfrak{p}^J\big)^K = \mathrm{Pol}_{n,m}^{U(n)}$, then

$$\text{(3.11)} \qquad \Big(\Theta_{n,m}(P)f\Big)(gK^J) = \left[P\left(\frac{\partial}{\partial t_\alpha}\right) f\left(g\exp\left(\sum_{\alpha=1}^{N_\star} t_\alpha \eta_\alpha\right) K^J\right)\right]_{(t_\alpha)=0},$$

where $g \in G^J$ and $f \in C^\infty(\mathbb{H}_{n,m})$. In general, it is hard to express $\Theta_{n,m}(P)$ explicitly for a polynomial $P \in \mathrm{Pol}\big(\mathfrak{p}^J\big)^K$. We refer to [10], p. 287.

We present the following *basic* $U(n)$-invariant polynomials in $\mathrm{Pol}_{n,m}^{U(n)}$.

$$\text{(3.12)} \qquad q_j(\omega,z) = \mathrm{tr}\big((\omega\,\overline{\omega})^{j+1}\big), \quad 0 \le j \le n-1,$$

$$\text{(3.13)} \qquad \alpha_{kp}^{(j)}(\omega,z) = \mathrm{Re}\,\big(z\,(\overline{\omega}\omega)^j\,{}^t\overline{z}\big)_{kp}, \quad 0 \le j \le n-1,\ 1 \le k \le p \le m,$$

$$\text{(3.14)} \qquad \beta_{lq}^{(j)}(\omega,z) = \mathrm{Im}\,\big(z\,(\overline{\omega}\omega)^j\,{}^t\overline{z}\big)_{lq}, \quad 0 \le j \le n-1,\ 1 \le l < q \le m,$$

$$\text{(3.15)} \qquad f_{kp}^{(j)}(\omega,z) = \mathrm{Re}\,(z\,(\overline{\omega}\omega)^j\,\overline{\omega}\,{}^tz)_{kp}, \quad 0 \le j \le n-1,\ 1 \le k \le p \le m,$$

$$\text{(3.16)} \qquad g_{kp}^{(j)}(\omega,z) = \mathrm{Im}\,(z\,(\overline{\omega}\omega)^j\,\overline{\omega}\,{}^tz\,)_{kp}, \quad 0 \le j \le n-1,\ 1 \le k \le p \le m,$$

where $\omega \in T_n$ and $z \in \mathbb{C}^{(m,n)}$.

We present some interesting $U(n)$-invariants. For an $m\times m$ matrix S, we define the following invariant polynomials in $\mathrm{Pol}_{n,m}^{U(n)}$:

$$m_{j;S}^{(1)}(\omega,z)=\ \mathrm{Re}\left(\mathrm{tr}\big(\omega\overline{\omega}+\,{}^tzS\overline{z}\,\big)^j\right),\quad 1\le j\le n, \tag{3.17}$$

$$m_{j;S}^{(2)}(\omega,z)=\ \mathrm{Im}\left(\mathrm{tr}\big(\omega\overline{\omega}+\,{}^tzS\overline{z}\,\big)^j\right),\quad 1\le j\le n, \tag{3.18}$$

$$q_{k;S}^{(1)}(\omega,z)=\ \mathrm{Re}\left(\mathrm{tr}\big((\,{}^tz\,S\,\overline{z})^k\big)\right),\quad 1\le k\le m, \tag{3.19}$$

$$q_{k;S}^{(2)}(\omega,z)=\ \mathrm{Im}\left(\mathrm{tr}\big((\,{}^tz\,S\,\overline{z})^k\big)\right),\quad 1\le k\le m, \tag{3.20}$$

$$\theta_{i,k,j;S}^{(1)}(\omega,z)=\ \mathrm{Re}\left(\mathrm{tr}\big((\omega\overline{\omega})^i\,(\,{}^tz\,S\,\overline{z})^k\,(\omega\overline{\omega}+\,{}^tz\,S\,\overline{z}\,)^j\,\big)\right), \tag{3.21}$$

$$\theta_{i,k,j;S}^{(2)}(\omega,z)=\ \mathrm{Im}\left(\mathrm{tr}\big((\omega\overline{\omega})^i\,(\,{}^tz\,S\,\overline{z})^k\,(\omega\overline{\omega}+\,{}^tz\,S\,\overline{z}\,)^j\,\big)\right), \tag{3.22}$$

where $1\le i,j\le n$ and $1\le k\le m$.

We define the following $U(n)$-invariant polynomials in $\mathrm{Pol}_{n,m}^{U(n)}$.

$$r_{jk}^{(1)}(\omega,z)=\ \mathrm{Re}\left(\det\big((\omega\overline{\omega})^j\,(\,{}^tz\overline{z})^k\,\big)\right),\quad 1\le j\le n,\ 1\le k\le m, \tag{3.23}$$

$$r_{jk}^{(2)}(\omega,z)=\ \mathrm{Im}\left(\det\big((\omega\overline{\omega})^j\,(\,{}^tz\overline{z})^k\,\big)\right),\quad 1\le j\le n,\ 1\le k\le m. \tag{3.24}$$

We propose the following natural problems.

Problem 1. Find a complete list of explicit generators of $\mathrm{Pol}_{n,m}^{U(n)}$.

Problem 2. Find all the relations among a set of generators of $\mathrm{Pol}_{n,m}^{U(n)}$.

Problem 3. Find an easy or effective way to express the images of the above invariant polynomials or generators of $\mathrm{Pol}_{n,m}^{U(n)}$ under the Helgason map $\Theta_{n,m}$ explicitly.

Problem 4. Decompose $\mathrm{Pol}_{n,m}$ into $U(n)$-irreducibles.

Problem 5. Find a complete list of explicit generators of the algebra $\mathbb{D}(\mathbb{H}_{n,m})$. Or construct explicit G^J-invariant differential operators on $\mathbb{H}_{n,m}$.

Problem 6. Find all the relations among a set of generators of $\mathbb{D}(\mathbb{H}_{n,m})$.

Problem 7. Is $\mathrm{Pol}_{n,m}^{U(n)}$ finitely generated ? Is $\mathbb{D}(\mathbb{H}_{n,m})$ finitely generated ?

Quite recently Minoru Itoh [12] solved Problem 1 and Problem 7.

Theorem 3.2. $\mathrm{Pol}_{n,m}^{U(n)}$ *is generated by*

$$q_j(\omega,z),\ \alpha_{kp}^{(j)}(\omega,z),\ \beta_{lq}^{(j)}(\omega,z),\ f_{kp}^{(j)}(\omega,z)\ \textit{and}\ g_{kp}^{(j)}(\omega,z),$$

where $0\le j\le n-1,\ 1\le k\le p\le m$ *and* $1\le l<q\le m$.

4. Examples of Explicit G^J-Invariant Differential Operators

In this section we give examples of explicit G^J-invariant differential operators on the Siegel-Jacobi space and the Siegel-Jacobi disk.

For $g = \big(M, (\lambda, \mu; \kappa)\big) \in G^J$ with $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$ and $(\Omega, Z) \in \mathbb{H}_{n,m}$, we set

$$\begin{aligned} \Omega_* &= M\cdot\Omega = X_* + i\,Y_*, \quad X_*, Y_* \text{ real}, \\ Z_* &= (Z + \lambda\Omega + \mu)(C\Omega + D)^{-1} = U_* + i\,V_*, \quad U_*, V_* \text{ real}. \end{aligned}$$

For a coordinate $(\Omega, Z) \in \mathbb{H}_{n,m}$ with $\Omega = (\omega_{\mu\nu})$ and $Z = (z_{kl})$, we put $d\Omega$, $d\overline{\Omega}$, $\frac{\partial}{\partial\Omega}$, $\frac{\partial}{\partial\overline{\Omega}}$ as before and set

$$\begin{aligned} Z &= U + iV, \quad U = (u_{kl}), \quad V = (v_{kl}) \text{ real}, \\ dZ &= (dz_{kl}), \quad d\overline{Z} = (d\overline{z}_{kl}), \end{aligned}$$

$$\frac{\partial}{\partial Z} = \begin{pmatrix} \frac{\partial}{\partial z_{11}} & \cdots & \frac{\partial}{\partial z_{m1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial z_{1n}} & \cdots & \frac{\partial}{\partial z_{mn}} \end{pmatrix}, \quad \frac{\partial}{\partial \overline{Z}} = \begin{pmatrix} \frac{\partial}{\partial \overline{z}_{11}} & \cdots & \frac{\partial}{\partial \overline{z}_{m1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial \overline{z}_{1n}} & \cdots & \frac{\partial}{\partial \overline{z}_{mn}} \end{pmatrix}.$$

Then we can show that

$$d\Omega_* = {}^t(C\Omega + D)^{-1}\, d\Omega (C\Omega + D)^{-1}, \tag{4.1}$$

$$dZ_* = dZ(C\Omega + D)^{-1} + \big\{\lambda - (Z + \lambda\Omega + \mu)(C\Omega + D)^{-1}C\big\}\, d\Omega (C\Omega + D)^{-1}, \tag{4.2}$$

$$\frac{\partial}{\partial\Omega_*} = (C\Omega + D)\, {}^t\!\left\{(C\Omega + D)\frac{\partial}{\partial\Omega}\right\} + (C\Omega + D)\, {}^t\!\left\{\big(C\,{}^tZ + C\,{}^t\mu - D\,{}^t\lambda\big)\, {}^t\!\left(\frac{\partial}{\partial Z}\right)\right\} \tag{4.3}$$

and

$$\frac{\partial}{\partial Z_*} = (C\Omega + D)\frac{\partial}{\partial Z}. \tag{4.4}$$

From [14, p. 33] or [20, p. 128], we know that

$$Y_* = {}^t(C\overline{\Omega} + D)^{-1} Y (C\Omega + D)^{-1} = {}^t(C\Omega + D)^{-1} Y (C\overline{\Omega} + D)^{-1}. \tag{4.5}$$

Using Formulas (4.1), (4.2) and (4.5), the author [29] proved that for any two positive real numbers A and B,

$$\begin{aligned} ds^2_{n,m;A,B} \;&=\; A\,\mathrm{tr}\Big(Y^{-1}d\Omega\,Y^{-1}d\overline{\Omega}\Big) \\ &\quad + B\,\Big\{\mathrm{tr}\Big(Y^{-1}\,{}^tV\,V\,Y^{-1}d\Omega\,Y^{-1}d\overline{\Omega}\Big) + \mathrm{tr}\Big(Y^{-1}\,{}^t(dZ)\,d\overline{Z}\Big) \\ &\qquad - \mathrm{tr}\Big(V\,Y^{-1}d\Omega\,Y^{-1}\,{}^t(d\overline{Z})\Big) - \mathrm{tr}\Big(V\,Y^{-1}d\overline{\Omega}\,Y^{-1}\,{}^t(dZ)\,\Big)\Big\} \end{aligned}$$

is a Riemannian metric on $\mathbb{H}_{n,m}$ which is invariant under the action (1.2) of G^J.

The following lemma is very useful for computing the invariant differential operators. H. Maass [13] observed the following useful fact.

Lemma 4.1. *(a) Let A be an $m\times n$ matrix and B an $n\times l$ matrix. Assume that the entries of A commute with the entries of B. Then ${}^t(AB) = {}^tB\,{}^tA$.*
(b) Let A, B and C be a $k\times l$, an $n\times m$ and an $m\times l$ matrix respectively. Assume that the entries of A commute with the entries of B. Then

$${}^t(A\,{}^t(BC)) = B\,{}^t(A\,{}^tC).$$

Proof. The proof follows immediately from the direct computation. □

Using Formulas (4.3), (4.4), (4.5) and Lemma 4.1, the author [29] proved that the following differential operators $\mathbb{M}_1$ and $\mathbb{M}_2$ on $\mathbb{H}_{n,m}$ defined by

$$\mathbb{M}_1 = \mathrm{tr}\left(Y\,\frac{\partial}{\partial Z}\,{}^t\!\left(\frac{\partial}{\partial \overline{Z}}\right)\right) \tag{4.6}$$

and

$$\begin{aligned} \mathbb{M}_2 \;&=\; \mathrm{tr}\left(Y\,{}^t\!\left(Y\frac{\partial}{\partial\overline{\Omega}}\right)\frac{\partial}{\partial\Omega}\right) + \mathrm{tr}\left(VY^{-1}\,{}^tV\,{}^t\!\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial Z}\right) \\ &\quad + \mathrm{tr}\left(V\,{}^t\!\left(Y\frac{\partial}{\partial\overline{\Omega}}\right)\frac{\partial}{\partial Z}\right) + \mathrm{tr}\left({}^tV\,{}^t\!\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial\Omega}\right) \end{aligned} \tag{4.7}$$

are invariant under the action (1.2) of G^J. The author [29] proved that for any two positive real numbers A and B, the following differential operator

$$\Delta_{n,m;A,B} = \frac{4}{A}\,\mathbb{M}_2 + \frac{4}{B}\mathbb{M}_1 \tag{4.8}$$

is the Laplacian of the G^J-invariant Riemannian metric $ds^2_{n,m;A,B}$.

Proposition 4.1. *The following differential operator* $\mathbb{K}$ *on* $\mathbb{H}_{n,m}$ *of degree* $2n$ *defined by*

$$\mathbb{K} = \det(Y)\, \det \left(\frac{\partial}{\partial Z}\, {}^t\!\left(\frac{\partial}{\partial \overline{Z}} \right) \right) \tag{4.9}$$

is invariant under the action (1.2) of G^J.

Proof. Let $\mathbb{K}_{M,(\lambda,\mu;\kappa)}$ denote the image of $\mathbb{K}$ under the transformation

$$(\Omega, Z) \longmapsto \big((M\cdot\Omega, (Z+\lambda\Omega+\mu)(C\Omega+D)^{-1}\big)$$

with $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R})$ and $(\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)}$. If f is a C^∞ function on $\mathbb{H}_{n,m}$, using (4.4), (4.5) and Lemma 4.1, we have

$$\begin{aligned}
\mathbb{K}_{M,(\lambda,\mu;\kappa)} f &= \det(Y)\,|\det(C\Omega+D)|^{-2}\, \det \left[(C\Omega+D) \frac{\partial}{\partial Z}\, {}^t\!\left\{ (C\overline{\Omega}+D) \frac{\partial f}{\partial \overline{Z}} \right\} \right] \\
&= \det(Y)\,|\det(C\Omega+D)|^{-2}\, \det \left[{}^t(C\Omega+D) \left\{ {}^t(C\overline{\Omega}+D) \left(\frac{\partial}{\partial Z}\, {}^t\!\left(\frac{\partial f}{\partial \overline{Z}} \right) \right) \right\} \right] \\
&= \det(Y)\,|\det(C\Omega+D)|^{-2}\, \det \left[(C\Omega+D) \frac{\partial}{\partial Z}\, {}^t\!\left(\frac{\partial f}{\partial \overline{Z}} \right) {}^t(C\overline{\Omega}+D) \right] \\
&= \det(Y)\, \det \left(\frac{\partial}{\partial Z}\, {}^t\!\left(\frac{\partial f}{\partial \overline{Z}} \right) \right) \\
&= \mathbb{K} f.
\end{aligned}$$

Since $M \in Sp(n,\mathbb{R})$ and $(\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)}$ are arbitrary, $\mathbb{K}$ is invariant under the action (1.2) of G^J. □

Proposition 4.2. *The following matrix-valued differential operator* $\mathbb{T}$ *on* $\mathbb{H}_{n,m}$ *defined by*

$$\mathbb{T} = {}^t\!\left(\frac{\partial}{\partial \overline{Z}} \right) Y \frac{\partial}{\partial Z} \tag{4.10}$$

is invariant under the action (1.2) of G^J.

Proof. Let $\mathbb{T}_{M,(\lambda,\mu;\kappa)}$ denote the image of $\mathbb{K}$ under the transformation

$$(\Omega, Z) \longmapsto \big((M\cdot\Omega, (Z+\lambda\Omega+\mu)(C\Omega+D)^{-1}\big)$$

with $M=\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in Sp(n,\mathbb{R})$ and $(\lambda,\mu;\kappa)\in H_{\mathbb{R}}^{(n,m)}$. If f is a C^∞ function on $\mathbb{H}_{n,m}$, according to (4.4), (4.5) and Lemma 4.1, we have

$$\begin{aligned} \mathbb{T}_{M,(\lambda,\mu;\kappa)}f &= {}^t\!\left((C\overline{\Omega}+D)\frac{\partial}{\partial\overline{Z}}\right)\,{}^t(C\overline{\Omega}+D)^{-1}Y(C\Omega+D)^{-1}(C\Omega+D)\frac{\partial f}{\partial Z} \\ &= {}^t\!\left(\frac{\partial}{\partial\overline{Z}}\right)Y\frac{\partial f}{\partial Z} \\ &= \mathbb{T}f. \end{aligned}$$

Since $M\in Sp(n,\mathbb{R})$ and $(\lambda,\mu;\kappa)\in H_{\mathbb{R}}^{(n,m)}$ are arbitrary, $\mathbb{T}$ is invariant under the action (1.2) of G^J. □

Corollary 4.1. *Each (k,l)-entry $\mathbb{T}_{kl}$ of $\mathbb{T}$ given by*

$$\mathbb{T}_{kl}=\sum_{i,j=1}^n y_{ij}\,\frac{\partial^2}{\partial\overline{z}_{ki}\partial z_{lj}},\quad 1\le k,l\le m \tag{4.11}$$

is an element of $\mathbb{D}\big(\mathbb{H}_{n,m}\big)$.

Proof. It follows immediately from Proposition 4.2. □

Now we consider invariant differential operators on the Siegel-Jacobi disk. Let

$$\mathbb{D}_n=\left\{W\in\mathbb{C}^{(n,n)}\mid W={}^tW,\ I_n-\overline{W}W>0\,\right\}$$

be the generalized unit disk.

For brevity, we write $\mathbb{D}_{n,m}:=\mathbb{D}_n\times\mathbb{C}^{(m,n)}$. For a coordinate $(W,\eta)\in\mathbb{D}_{n,m}$ with $W=(w_{\mu\nu})\in\mathbb{D}_n$ and $\eta=(\eta_{kl})\in\mathbb{C}^{(m,n)}$, we put

$$\begin{aligned} dW &= (dw_{\mu\nu}), \qquad d\overline{W} = (d\overline{w}_{\mu\nu}), \\ d\eta &= (d\eta_{kl}), \qquad d\overline{\eta} = (d\overline{\eta}_{kl}) \end{aligned}$$

and

$$\frac{\partial}{\partial W}=\left(\frac{1+\delta_{\mu\nu}}{2}\,\frac{\partial}{\partial w_{\mu\nu}}\right),\quad \frac{\partial}{\partial\overline{W}}=\left(\frac{1+\delta_{\mu\nu}}{2}\,\frac{\partial}{\partial\overline{w}_{\mu\nu}}\right),$$

$$\frac{\partial}{\partial\eta}=\begin{pmatrix} \frac{\partial}{\partial\eta_{11}} & \cdots & \frac{\partial}{\partial\eta_{m1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial\eta_{1n}} & \cdots & \frac{\partial}{\partial\eta_{mn}} \end{pmatrix},\quad \frac{\partial}{\partial\overline{\eta}}=\begin{pmatrix} \frac{\partial}{\partial\overline{\eta}_{11}} & \cdots & \frac{\partial}{\partial\overline{\eta}_{m1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial\overline{\eta}_{1n}} & \cdots & \frac{\partial}{\partial\overline{\eta}_{mn}} \end{pmatrix}.$$

We can identify an element $g = (M, (\lambda, \mu; \kappa))$ of G^J, $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$ with the element

$$\begin{pmatrix} A & 0 & B & A\,{}^t\mu - B\,{}^t\lambda \\ \lambda & I_m & \mu & \kappa \\ C & 0 & D & C\,{}^t\mu - D\,{}^t\lambda \\ 0 & 0 & 0 & I_m \end{pmatrix}$$

of $Sp(m + n, \mathbb{R})$.

We set

$$T_* = \frac{1}{\sqrt{2}} \begin{pmatrix} I_{m+n} & I_{m+n} \\ iI_{m+n} & -iI_{m+n} \end{pmatrix}.$$

We now consider the group G^J_* defined by

$$G^J_* := T_*^{-1} G^J T_*.$$

If $g = (M, (\lambda, \mu; \kappa)) \in G^J$ with $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$, then $T_*^{-1} g T_*$ is given by

$$T_*^{-1} g T_* = \begin{pmatrix} P_* & Q_* \\ \overline{Q}_* & \overline{P}_* \end{pmatrix}, \tag{4.12}$$

where

$$P_* = \begin{pmatrix} P & \frac{1}{2}\left\{Q\ {}^t(\lambda + i\mu) - P\ {}^t(\lambda - i\mu)\right\} \\ \frac{1}{2}(\lambda + i\mu) & I_h + i\frac{\kappa}{2} \end{pmatrix},$$

$$Q_* = \begin{pmatrix} Q & \frac{1}{2}\left\{P\ {}^t(\lambda - i\mu) - Q\ {}^t(\lambda + i\mu)\right\} \\ \frac{1}{2}(\lambda - i\mu) & -i\frac{\kappa}{2} \end{pmatrix},$$

and P, Q are given by the formulas

$$P = \frac{1}{2}\ \{(A + D) + i\,(B - C)\} \tag{4.13}$$

and

$$Q = \frac{1}{2}\ \{(A - D) - i\,(B + C)\}. \tag{4.14}$$

From now on, we write

$$\left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, \left(\frac{1}{2}(\lambda + i\mu),\ \frac{1}{2}(\lambda - i\mu);\ -i\frac{\kappa}{2} \right) \right) := \begin{pmatrix} P_* & Q_* \\ \overline{Q}_* & \overline{P}_* \end{pmatrix}.$$

In other words, we have the relation

$$T_*^{-1} \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, (\lambda, \mu; \kappa) \right) T_* = \left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, \left(\frac{1}{2}(\lambda + i\mu),\ \frac{1}{2}(\lambda - i\mu);\ -i\frac{\kappa}{2} \right) \right).$$

Let

$$H_{\mathbb{C}}^{(n,m)} := \left\{ (\xi, \eta\,; \zeta) \mid \xi, \eta \in \mathbb{C}^{(m,n)},\ \zeta \in \mathbb{C}^{(m,m)},\ \zeta + \eta\,{}^t\xi \text{ symmetric} \right\}$$

be the complex Heisenberg group endowed with the following multiplication

$$(\xi, \eta\,; \zeta) \circ (\xi', \eta'; \zeta') := (\xi + \xi', \eta + \eta'\,; \zeta + \zeta' + \xi\,{}^t\eta' - \eta\,{}^t\xi')).$$

We define the semidirect product

$$SL(2n,\mathbb{C}) \ltimes H_{\mathbb{C}}^{(n,m)}$$

endowed with the following multiplication

$$\begin{aligned}&\left(\begin{pmatrix} P & Q \\ R & S\end{pmatrix}, (\xi,\eta\,;\zeta)\right)\cdot\left(\begin{pmatrix} P' & Q' \\ R' & S'\end{pmatrix}, (\xi',\eta';\zeta')\right)\\ = &\left(\begin{pmatrix} P & Q \\ R & S\end{pmatrix}\begin{pmatrix} P' & Q' \\ R' & S'\end{pmatrix},\, (\tilde{\xi}+\xi',\tilde{\eta}+\eta';\zeta+\zeta'+\tilde{\xi}\,{}^t\eta'-\tilde{\eta}\,{}^t\xi')\right),\end{aligned}$$

where $\tilde{\xi}=\xi P'+\eta R'$ and $\tilde{\eta}=\xi Q'+\eta S'$.

If we identify $H_{\mathbb{R}}^{(n,m)}$ with the subgroup

$$\left\{(\xi,\overline{\xi};i\kappa)\mid \xi\in\mathbb{C}^{(m,n)},\ \kappa\in\mathbb{R}^{(m,m)}\right\}$$

of $H_{\mathbb{C}}^{(n,m)}$, we have the following inclusion

$$G_*^J\subset SU(n,n)\ltimes H_{\mathbb{R}}^{(n,m)}\subset SL(2n,\mathbb{C})\ltimes H_{\mathbb{C}}^{(n,m)}.$$

We define the mapping $\Theta: G^J\longrightarrow G_*^J$ by

$$\Theta\left(\begin{pmatrix} A & B \\ C & D\end{pmatrix}, (\lambda,\mu;\kappa)\right) := \left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P}\end{pmatrix}, \left(\frac{1}{2}(\lambda+i\mu),\,\frac{1}{2}(\lambda-i\mu);\,-i\frac{\kappa}{2}\right)\right), \tag{4.15}$$

where P and Q are given by (4.13) and (4.14). We can see that if $g_1,g_2\in G^J$, then $\Theta(g_1g_2)=\Theta(g_1)\Theta(g_2)$.

According to [26, p. 250], G_*^J is of the Harish-Chandra type (cf. [17, p. 118]). Let

$$g_*=\left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P}\end{pmatrix}, (\lambda,\mu;\,\kappa)\right)$$

be an element of G_*^J. Since the Harish-Chandra decomposition of an element $\begin{pmatrix} P & Q \\ R & S\end{pmatrix}$ in $SU(n,n)$ is given by

$$\begin{pmatrix} P & Q \\ R & S\end{pmatrix}=\begin{pmatrix} I_n & QS^{-1} \\ 0 & I_n\end{pmatrix}\begin{pmatrix} P-QS^{-1}R & 0 \\ 0 & S\end{pmatrix}\begin{pmatrix} I_n & 0 \\ S^{-1}R & I_n\end{pmatrix},$$

the P_*^+-component of the following element

$$g_*\cdot\left(\begin{pmatrix} I_n & W \\ 0 & I_n\end{pmatrix}, (0,\eta;0)\right),\quad W\in\mathbb{D}_n$$

of $SL(2n,\mathbb{C})\ltimes H_{\mathbb{C}}^{(n,m)}$ is given by

$$\left(\begin{pmatrix} I_n & (PW+Q)(\overline{Q}W+\overline{P})^{-1} \\ 0 & I_n\end{pmatrix},\, \left(0,\,(\eta+\lambda W+\mu)(\overline{Q}W+\overline{P})^{-1}\,;0\right)\right). \tag{4.16}$$

We can identify $\mathbb{D}_{n,m}$ with the subset

$$\left\{ \left(\begin{pmatrix} I_n & W \\ 0 & I_n \end{pmatrix}, (0,\eta;0) \right) \,\Big|\, W \in \mathbb{D}_n,\ \eta \in \mathbb{C}^{(m,n)} \right\}$$

of the complexification of G^J_*. Indeed, $\mathbb{D}_{n,m}$ is embedded into P^+_* given by

$$P^+_* = \left\{ \left(\begin{pmatrix} I_n & W \\ 0 & I_n \end{pmatrix}, (0,\eta;0) \right) \,\Big|\, W = {}^tW \in \mathbb{C}^{(n,n)},\ \eta \in \mathbb{C}^{(m,n)} \right\}.$$

This is a generalization of the Harish-Chandra embedding (cf. [17, p. 119]). Then we get the *natural transitive action* of G^J_* on $\mathbb{D}_{n,m}$ defined by

$$\begin{aligned} &\left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, \left(\xi, \overline{\xi};\, i\kappa\right) \right) \cdot (W,\eta) \\ &= \Big((PW+Q)(\overline{Q}W+\overline{P})^{-1}, (\eta+\xi W+\overline{\xi})(\overline{Q}W+\overline{P})^{-1} \Big), \end{aligned} \tag{4.17}$$

where $\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \in G_*,\ \xi \in \mathbb{C}^{(m,n)},\ \kappa \in \mathbb{R}^{(m,m)}$ and $(W,\eta) \in \mathbb{D}_{n,m}$.

The author [30] proved that the action (1.2) of G^J on $\mathbb{H}_{n,m}$ is compatible with the action (4.17) of G^J_* on $\mathbb{D}_{n,m}$ through a *partial Cayley transform* $\Phi : \mathbb{D}_{n,m} \longrightarrow \mathbb{H}_{n,m}$ defined by

$$\Phi(W,\eta) := \Big(i(I_n+W)(I_n-W)^{-1},\, 2\,i\,\eta\,(I_n-W)^{-1} \Big). \tag{4.18}$$

In other words, if $g_0 \in G^J$ and $(W,\eta) \in \mathbb{D}_{n,m}$,

$$g_0 \cdot \Phi(W,\eta) = \Phi(g_* \cdot (W,\eta)), \tag{4.19}$$

where $g_* = T_*^{-1} g_0 T_*$. Φ is a biholomorphic mapping of $\mathbb{D}_{n,m}$ onto $\mathbb{H}_{n,m}$ which gives the partially bounded realization of $\mathbb{H}_{n,m}$ by $\mathbb{D}_{n,m}$. The inverse of Φ is

$$\Phi^{-1}(\Omega, Z) = \Big((\Omega - iI_n)(\Omega + iI_n)^{-1},\, Z(\Omega + iI_n)^{-1} \Big).$$

For $(W,\eta) \in \mathbb{D}_{n,m}$, we write

$$(\Omega, Z) := \Phi(W,\eta).$$

Thus

$$\Omega = i(I_n+W)(I_n-W)^{-1}, \qquad Z = 2\,i\,\eta\,(I_n-W)^{-1}. \tag{4.20}$$

Since

$$d(I_n-W)^{-1} = (I_n-W)^{-1} dW\,(I_n-W)^{-1}$$

and

$$I_n + (I_n+W)(I_n-W)^{-1} = 2\,(I_n-W)^{-1},$$

we get the following formulas from (4.20)

$$\begin{aligned}
(4.21)\qquad Y &= \frac{1}{2\,i}\,(\Omega-\overline{\Omega}\,) = (I_n-W)^{-1}(I_n-W\overline{W}\,)(I_n-\overline{W}\,)^{-1},\\
(4.22)\qquad V &= \frac{1}{2\,i}\,(Z-\overline{Z}\,) = \eta\,(I_n-W)^{-1}+\overline{\eta}\,(I_n-\overline{W}\,)^{-1},\\
(4.23)\qquad d\Omega &= 2\,i\,(I_n-W)^{-1}dW\,(I_n-W)^{-1},\\
(4.24)\qquad dZ &= 2\,i\Big\{d\eta+\eta\,(I_n-W)^{-1}dW\Big\}(I_n-W)^{-1}.
\end{aligned}$$

Using Formulas (4.18), (4.20)-(4.24), the author [31] proved that for any two positive real numbers A and B, the following metric $d\tilde{s}^2_{n,m;A,B}$ defined by

$$\begin{aligned}
ds^2_{\mathbb{D}_{n,m};A,B} = \;& 4\,A\,\mathrm{tr}\Big((I_n-W\overline{W})^{-1}dW(I_n-\overline{W}W)^{-1}d\overline{W}\Big)\\
&+4\,B\,\Big\{\mathrm{tr}\Big((I_n-W\overline{W})^{-1}\,{}^t(d\eta)\,\beta\Big)\\
&\quad+\mathrm{tr}\Big((\eta\overline{W}-\overline{\eta})(I_n-W\overline{W})^{-1}dW(I_n-\overline{W}W)^{-1}\,{}^t(d\overline{\eta})\Big)\\
&\quad+\mathrm{tr}\Big((\overline{\eta}W-\eta)(I_n-\overline{W}W)^{-1}d\overline{W}(I_n-W\overline{W})^{-1}\,{}^t(d\eta)\Big)\\
&\quad-\mathrm{tr}\Big((I_n-W\overline{W})^{-1}\,{}^t\eta\,\eta\,(I_n-\overline{W}W)^{-1}\overline{W}dW(I_n-\overline{W}W)^{-1}d\overline{W}\Big)\\
&\quad-\mathrm{tr}\Big(W(I_n-\overline{W}W)^{-1}\,{}^t\overline{\eta}\,\overline{\eta}\,(I_n-W\overline{W})^{-1}dW(I_n-\overline{W}W)^{-1}d\overline{W}\Big)\\
&\quad+\mathrm{tr}\Big((I_n-W\overline{W})^{-1t}\eta\,\overline{\eta}\,(I_n-W\overline{W})^{-1}dW(I_n-\overline{W}W)^{-1}d\overline{W}\Big)\\
&\quad+\mathrm{tr}\Big((I_n-\overline{W})^{-1}\,{}^t\overline{\eta}\,\eta\,\overline{W}\,(I_n-W\overline{W})^{-1}dW(I_n-\overline{W}W)^{-1}d\overline{W}\Big)\\
&\quad+\mathrm{tr}\Big((I_n-\overline{W})^{-1}(I_n-W)(I_n-\overline{W}W)^{-1}\,{}^t\overline{\eta}\,\eta\,(I_n-\overline{W}W)^{-1}\\
&\qquad\qquad\times(I_n-\overline{W})(I_n-W)^{-1}dW(I_n-\overline{W}W)^{-1}d\overline{W}\Big)\\
&\quad-\mathrm{tr}\Big((I_n-W\overline{W})^{-1}(I_n-W)(I_n-\overline{W})^{-1}\,{}^t\overline{\eta}\,\eta\,(I_n-W)^{-1}\\
&\qquad\qquad\times dW(I_n-\overline{W}W)^{-1}d\overline{W}\Big)\Big\}
\end{aligned}$$

is a Riemannian metric on $\mathbb{D}_{n,m}$ which is invariant under the action (4.17) of the Jacobi group G^J_*.

We note that if $n = m = 1$ and $A = B = 1$, we get

$$\begin{aligned}\frac{1}{4}\,ds^2_{\mathbb{D}_{1,1};1,1} &= \frac{dW\,d\overline{W}}{(1-|W|^2)^2} + \frac{1}{(1-|W|^2)}\,d\eta\,d\overline{\eta} \\ &+ \frac{(1+|W|^2)|\eta|^2 - \overline{W}\eta^2 - W\overline{\eta}^2}{(1-|W|^2)^3}\,dW\,d\overline{W} \\ &+ \frac{\eta\overline{W}-\overline{\eta}}{(1-|W|^2)^2}\,dW d\overline{\eta} + \frac{\overline{\eta}W-\eta}{(1-|W|^2)^2}\,d\overline{W} d\eta.\end{aligned}$$

From the formulas (4.20), (4.23) and (4.24), we get

$$\frac{\partial}{\partial\Omega} = \frac{1}{2\,i}\,(I_n - W)\left[{}^t\!\left\{ (I_n - W)\,\frac{\partial}{\partial W} \right\} - {}^t\!\left\{ {}^t\eta\, {}^t\!\left(\frac{\partial}{\partial \eta} \right) \right\} \right] \tag{4.25}$$

and

$$\frac{\partial}{\partial Z} = \frac{1}{2\,i}\,(I_n - W)\frac{\partial}{\partial \eta}. \tag{4.26}$$

Using Formulas (4.20)-(4.22), (4.25), (4.26) and Lemma 4.1, the author [31] proved that the following differential operators $\mathbb{S}_1$ and $\mathbb{S}_2$ on $\mathbb{D}_{n,m}$ defined by

$$\mathbb{S}_1 = \sigma\left((I_n - \overline{W}W)\frac{\partial}{\partial\eta}\, {}^t\!\left(\frac{\partial}{\partial\overline{\eta}}\right)\right)$$

and

$$\begin{aligned}\mathbb{S}_2 &= \operatorname{tr}\left((I_n - W\overline{W})\, {}^t\!\left((I_n - W\overline{W})\frac{\partial}{\partial\overline{W}}\right)\frac{\partial}{\partial W}\right) \\ &+ \operatorname{tr}\left({}^t(\eta - \overline{\eta}\,W)\, {}^t\!\left(\frac{\partial}{\partial\overline{\eta}}\right)(I_n - \overline{W}W)\frac{\partial}{\partial W}\right) \\ &+ \operatorname{tr}\left((\overline{\eta} - \eta\,\overline{W})\, {}^t\!\left((I_n - W\overline{W})\frac{\partial}{\partial\overline{W}}\right)\frac{\partial}{\partial\eta}\right) \\ &- \operatorname{tr}\left(\eta\overline{W}(I_n - W\overline{W})^{-1}\, {}^t\eta\, {}^t\!\left(\frac{\partial}{\partial\overline{\eta}}\right)(I_n - \overline{W}W)\frac{\partial}{\partial\eta}\right) \\ &- \operatorname{tr}\left(\overline{\eta}W(I_n - \overline{W}W)^{-1}\, {}^t\overline{\eta}\, {}^t\!\left(\frac{\partial}{\partial\overline{\eta}}\right)(I_n - \overline{W}W)\frac{\partial}{\partial\eta}\right) \\ &+ \operatorname{tr}\left(\overline{\eta}(I_n - W\overline{W})^{-1}\, {}^t\eta\, {}^t\!\left(\frac{\partial}{\partial\overline{\eta}}\right)(I_n - \overline{W}W)\frac{\partial}{\partial\eta}\right) \\ &+ \operatorname{tr}\left(\eta\,\overline{W}W(I_n - \overline{W}W)^{-1}\, {}^t\overline{\eta}\, {}^t\!\left(\frac{\partial}{\partial\overline{\eta}}\right)(I_n - \overline{W}W)\frac{\partial}{\partial\eta}\right)\end{aligned}$$

are invariant under the action (4.17) of G_*^J. The author also proved that

$$\Delta_{\mathbb{D}_{n,m};A,B} := \frac{1}{A}\,\mathbb{S}_2 + \frac{1}{B}\,\mathbb{S}_1 \tag{4.27}$$

is the Laplacian of the invariant metric $ds^2_{\mathbb{D}_{n,m};A,B}$ on $\mathbb{D}_{n,m}$ (cf. [31]).

Proposition 4.3. *The following differential operator on $\mathbb{D}_{n,m}$ defined by*

$$\mathbb{K}_{\mathbb{D}} = \det(I_n - \overline{W}W)\,\det\left(\frac{\partial}{\partial\eta}\,{}^t\!\left(\frac{\partial}{\partial\overline{\eta}}\right)\right) \tag{4.28}$$

is invariant under the action (4.17) of G_*^J on $\mathbb{D}_{n,m}$.

Proof. It follows from Proposition 4.1, Formulas (4.21), (4.26) and the fact that the action (1.2) of G^J on $\mathbb{H}_{n,m}$ is compatible with the action (4.17) of G_*^J on $\mathbb{D}_{n,m}$ via the partial Cayley transform. □

Proposition 4.4. *The following matrix-valued differential operator on $\mathbb{D}_{n,m}$ defined by*

$$\mathbb{T}^{\mathbb{D}} := {}^t\!\left(\frac{\partial}{\partial\overline{\eta}}\right)(I_n - \overline{W}W)\frac{\partial}{\partial\eta} \tag{4.29}$$

is invariant under the action (4.17) of G_*^J on $\mathbb{D}_{n,m}$.

Proof. It follows from Proposition 4.2, Formulas (4.21), (4.26) and the fact that the action (1.2) of G^J on $\mathbb{H}_{n,m}$ is compatible with the action (4.17) of G_*^J on $\mathbb{D}_{n,m}$ via the partial Cayley transform. □

Corollary 4.2. *Each (k,l)-entry $\mathbb{T}^{\mathbb{D}}_{kl}$ of $\mathbb{T}^{\mathbb{D}}$ given by*

$$\mathbb{T}^{\mathbb{D}}_{kl} = \sum_{i,j=1}^{n}\left(\delta_{ij} - \sum_{r=1}^{n}\overline{w}_{ir}\,w_{jr}\right)\frac{\partial^2}{\partial\overline{\eta}_{ki}\partial\eta_{lj}}, \quad 1 \le k,l \le m \tag{4.30}$$

is a G_*^J-invariant differential operator on $\mathbb{D}_{n,m}$.

Proof. It follows immediately from Proposition 4.4. □

For two differential operators D_1 and D_2 on $\mathbb{H}_{n,m}$ or $\mathbb{D}_{n,m}$, we write

$$[D_1, D_2] := D_1D_2 - D_2D_1.$$

Then

$$\mathbb{M}_3 = [\mathbb{M}_1, \mathbb{M}_2] = \mathbb{M}_1\mathbb{M}_2 - \mathbb{M}_2\mathbb{M}_1 \tag{4.31}$$

is an invariant differential operator of degree three on $\mathbb{H}_{n,m}$ and

$$\mathbb{P}_{kl} = [\mathbb{K}, \mathbb{T}_{kl}] = \mathbb{K}\mathbb{T}_{kl} - \mathbb{T}_{kl}\mathbb{K}, \quad 1 \le k,l \le m \tag{4.32}$$

is an invariant differential operator of degree $2n+1$ on $\mathbb{H}_{n,m}$.

Similarly

$$\mathbb{S}_3 = [\mathbb{S}_1, \mathbb{S}_2] = \mathbb{S}_1\mathbb{S}_2 - \mathbb{S}_2\mathbb{S}_1 \tag{4.33}$$

is an invariant differential operator of degree three on $\mathbb{D}_{n,m}$ and

$$\mathbb{Q}_{kl} = [\mathbb{K}_{\mathbb{D}}, \mathbb{T}^{\mathbb{D}}_{kl}] = \mathbb{K}_{\mathbb{D}}\mathbb{T}^{\mathbb{D}}_{kl} - \mathbb{T}^{\mathbb{D}}_{kl}\mathbb{K}_{\mathbb{D}}, \quad 1 \le k, l \le m \tag{4.34}$$

is an invariant differential operator of degree $2n+1$ on $\mathbb{D}_{n,m}$.

Indeed it is very complicated and difficult at this moment to express the generators of the algebra of all G^J_*-invariant differential operators on $\mathbb{D}_{n,m}$ explicitly.

5. **The Case** $n = m = 1$

We consider the case $n = m = 1$. For a coordinate (w, ξ) in $T_{1,1} = \mathbb{C} \times \mathbb{C}$, we write $w = r + i\,s,\ \xi = \zeta + i\,\eta \in \mathbb{C},\ r, s, \zeta, \eta$ real. The author [27] proved that the algebra $\mathrm{Pol}_{1,1}^{U(1)}$ is generated by

$$\begin{aligned}
q(w,\xi) &= \frac{1}{4}\, w\,\overline{w} = \frac{1}{4}\big(r^2 + s^2\big),\\
\alpha(w,\xi) &= \xi\,\overline{\xi} = \zeta^2 + \eta^2,\\
\phi(w,\xi) &= \frac{1}{2}\,\mathrm{Re}\,\big(\xi^2\overline{w}\big) = \frac{1}{2}\, r\big(\zeta^2 - \eta^2\big) + s\,\zeta\eta,\\
\psi(w,\xi) &= \frac{1}{2}\,\mathrm{Im}\,\big(\xi^2\overline{w}\big) = \frac{1}{2}\, s\big(\eta^2 - \zeta^2\big) + r\,\zeta\eta.
\end{aligned}$$

In [27], using Formula (3.11) the author calculated explicitly the images

$$D_1 = \Theta_{1,1}(q), \quad D_2 = \Theta_{1,1}(\alpha), \quad D_3 = \Theta_{1,1}(\phi) \quad \text{and} \quad D_4 = \Theta_{1,1}(\psi)$$

of q, ξ, ϕ and ψ under the Helgason map $\Theta_{1,1}$. We can show that the algebra $\mathbb{D}(\mathbb{H}_{1,1})$ is generated by the following differential operators

$$\begin{aligned}
D_1 =\,& y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + v^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right)\\
&+ 2\,y\,v \left(\frac{\partial^2}{\partial x \partial u} + \frac{\partial^2}{\partial y \partial v} \right),
\end{aligned}$$

$$D_2 = y \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right),$$

$$\begin{aligned}
D_3 =\,& y^2\, \frac{\partial}{\partial y} \left(\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} \right) - 2y^2\, \frac{\partial^3}{\partial x \partial u \partial v}\\
&- \left(v\, \frac{\partial}{\partial v} + 1 \right) D_2
\end{aligned}$$

and

$$D_4 = y^2 \frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial v^2} - \frac{\partial^2}{\partial u^2}\right) - 2\,y^2 \frac{\partial^3}{\partial y \partial u \partial v} - v\,\frac{\partial}{\partial u} D_2,$$

where $\tau = x + iy$ and $z = u + iv$ with real variables x, y, u, v. Moreover, we have

$$D_1 D_2 - D_2 D_1 = 2\,y^2 \frac{\partial}{\partial y}\left(\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2}\right) - 4\,y^2 \frac{\partial^3}{\partial x \partial u \partial v} - 2\left(v\,\frac{\partial}{\partial v} D_2 + D_2\right).$$

In particular, the algebra $\mathbb{D}(\mathbb{H}_{1,1})$ is not commutative. We refer to [1, 27] for more detail.

Recently Hiroyuki Ochiai [15] (cf. [1]) proved the following results.

Theorem 5.1. *We have the following relation*

$$\phi^2 + \psi^2 = q\,\alpha^2. \tag{5.1}$$

This relation exhausts all the relations among the generators q, α, ϕ and ψ of $\mathrm{Pol}_{1,1}^{U(1)}$.

Theorem 5.2. *We have the following relations*

(*a*) $[D_1, D_2] = 2D_3$

(*b*) $[D_1, D_3] = 2D_1 D_2 - 2D_3$

(*c*) $[D_2, D_3] = -D_2^2$

(*d*) $[D_4, D_1] = 0$

(*e*) $[D_4, D_2] = 0$

(*f*) $[D_4, D_3] = 0$

(*g*) $D_3^2 + D_4^2 = D_2 D_1 D_2$

These seven relations exhaust all the relations among the generators D_1, D_2, D_3 and D_4 of $\mathbb{D}(\mathbb{H}_{1,1})$.

We can prove the following

Theorem 5.3. *The action of $U(1)$ on* $\mathrm{Pol}_{1,1}^{U(1)}$ *is not multiplicity-free.*

Finally we see that for the case $n = m = 1$, the seven problems proposed in Section 3 are completely solved.

Remark 5.1. *According to Theorem 5.2, we see that D_4 is a generator of the center of $\mathbb{D}(\mathbb{H}_{1,1})$. We observe that the Lapalcian*

$$\Delta_{1,1;A,B} = \frac{4}{A}\,D_1 + \frac{4}{B}\,D_2 \qquad \text{(see (4.8))}$$

of $(\mathbb{H}_{1,1}, ds^2_{1,1;A,B})$ does not belong to the center of $\mathbb{D}(\mathbb{H}_{1,1})$.

6. The Case $n = 1$ and m is arbitrary

Conley and Raum [5] found the $2m^2 + m + 1$ explicit generators of $\mathbb{D}(\mathbb{H}_{1,m})$ and the explicit one generator of the center of $\mathbb{D}(\mathbb{H}_{1,m})$. They also found the generators of the center of the universal enveloping algebra of $\mathfrak{U}(\mathfrak{g}^J)$ of the Jacobi Lie algebra $\mathfrak{g}^J$. The number of generators of the center of $\mathfrak{U}(\mathfrak{g}^J)$ is $1 + \frac{m(m+1)}{2}$.

According to Theorem 3.2, $\mathrm{Pol}_{1,m}^{U(1)}$ is generated by

$$q(w,\xi) = \mathrm{tr}\big(w\,\overline{w}\big), \tag{6.1}$$

$$\alpha_{kp}(w,\xi) = \mathrm{Re}\,\big(\xi\,{}^t\overline{\xi}\big)_{kp} = \mathrm{Re}\,(\xi_k\,\overline{\xi}_p), \quad 1 \le k \le p \le m, \tag{6.2}$$

$$\beta_{lq}(w,\xi) = \mathrm{Im}\,\big(\xi\,{}^t\overline{\xi}\big)_{lq} = \mathrm{Im}\,(\xi_l\,\overline{\xi}_q), \quad 1 \le l < q \le m, \tag{6.3}$$

$$f_{kp}(w,\xi) = \mathrm{Re}\,(\overline{w}\,\xi\,{}^t\xi)_{kp} = \mathrm{Re}\,(\overline{w}\,\xi_k\,\xi_p), \quad 1 \le k \le p \le m, \tag{6.4}$$

$$g_{kp}(w,\xi) = \mathrm{Im}\,(\overline{w}\,\xi\,{}^t\xi)_{kp} = \mathrm{Im}\,(\overline{w}\,\xi_k\,\xi_p), \quad 1 \le k \le p \le m, \tag{6.5}$$

where $w \in \mathbb{C}$ and $\xi \in \mathbb{C}^m$.

We let

$$w = r + i\,s \in \mathbb{C} \quad \text{and} \quad \xi = {}^t(\xi_1, \cdots, \xi_m) \in \mathbb{C}^m \text{ with } \xi_k = \zeta_k + i\,\eta_k,\ 1 \le k \le m,$$

where $r, s, \zeta_1, \eta_1, \cdots, \zeta_m, \eta_m$ are real. The invariants q, α_{kp}, β_{lq}, f_{kp} and g_{kp} are expressed in terms of r, s, ζ_k, η_l $(1 \le k, l \le m)$ as follows:

$$\begin{aligned}
q(w,\xi) &= r^2 + s^2, \\
\alpha_{kp}(w,\xi) &= \zeta_k\zeta_p + \eta_k\eta_p, \quad 1 \le k \le p \le m, \\
\beta_{lq}(w,\xi) &= \zeta_q\eta_l - \zeta_l\eta_q, \quad 1 \le l < q \le m, \\
f_{kp}(w,\xi) &= r(\zeta_k\zeta_p - \eta_k\eta_p) + s(\zeta_k\eta_p + \eta_k\zeta_p), \quad 1 \le k \le p \le m, \\
g_{kp}(w,\xi) &= r(\zeta_k\eta_p + \eta_k\zeta_p) - s(\zeta_k\zeta_p - \eta_k\eta_p), \quad 1 \le k \le p \le m.
\end{aligned}$$

Theorem 6.1. *The $\frac{m(m+1)}{2}$ relations*

$$f_{kp}^2 + g_{kp}^2 = q\,\alpha_{kk}\,\alpha_{pp}, \quad 1 \le k \le p \le m \tag{6.6}$$

exhaust all the relations among a complete set of generators q, α_{kp}, β_{lq}, f_{kp} and g_{kp} of $\mathrm{Pol}_{1,m}^{U(1)}$ with $1 \le k \le p \le m$ and $1 \le l < q \le m$.

Theorem 6.2. *The action of $U(1)$ on $Pol_{1,m}$ is not multiplicity-free. In fact, if*

$$\mathrm{Pol}_{1,m} = \sum_{\sigma \in \widehat{U(1)}} m_\sigma \, \sigma,$$

then $m_\sigma = \infty$.

Problem 1, Problem 2, Problem 4, Problem 5 and Problem 7 were solved. Problem 3 can be handled. Finally Problem 6 is unsolved in the case that $n = 1$ and m is arbitrary.

7. Final Remarks

Using G^J-invariant differential operators on the Siegel-Jacobi space, we introduce a notion of Maass-Jacobi forms.

Definition 7.1. *Let*

$$\Gamma_{n,m} := Sp(n, \mathbb{Z}) \ltimes H_{\mathbb{Z}}^{(n,m)}$$

be the discrete subgroup of G^J, where

$$H_{\mathbb{Z}}^{(n,m)} = \left\{ (\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(n,m)} \mid \lambda, \mu, \kappa \text{ are integral} \right\}.$$

A smooth function $f : \mathbb{H}_{n,m} \longrightarrow \mathbb{C}$ is called a Maass-Jacobi form *on $\mathbb{H}_{n,m}$ if f satisfies the following conditions (MJ1)-(MJ3):*

(MJ1) *f is invariant under $\Gamma_{n,m}$.*
(MJ2) *f is an eigenfunction of the Laplacian $\Delta_{n,m;A,B}$ (cf. Formula (4.8)).*
(MJ3) *f has a polynomial growth, that is, there exist a constant $C > 0$ and a positive integer N such that*

$$|f(X + iY, Z)| \leq C \, |p(Y)|^N \quad \text{as } \det Y \longrightarrow \infty,$$

where $p(Y)$ is a polynomial in $Y = (y_{ij})$.

Remark 7.1. *Let $\mathbb{D}_*$ be a commutative subalgebra of $\mathbb{D}(\mathbb{H}_{n,m})$ containing the Laplacian $\Delta_{n,m;A,B}$. We say that a smooth function $f : \mathbb{H}_{n,m} \longrightarrow \mathbb{C}$ is a Maass-Jacobi form with respect to $\mathbb{D}_*$ if f satisfies the conditions $(MJ1)$, $(MJ2)_*$ and $(MJ3)$: the condition $(MJ2)_*$ is given by*

$(MJ2)_*$ *f is an eigenfunction of any invariant differential operator in $\mathbb{D}_*$.*

It is natural to propose the following problems.

Problem A : Find all the eigenfunctions of $\Delta_{n,m;A,B}$.

Problem B : Construct Maass-Jacobi forms.

If we find a *nice* eigenfunction ϕ of the Laplacian $\Delta_{n,m;A,B}$, we can construct a Maass-Jacobi form f_ϕ on $\mathbb{H}_{n,m}$ in the usual way defined by

$$f_\phi(\Omega,Z) := \sum_{\gamma\in\Gamma^\infty_{n,m}\backslash\Gamma_{n,m}} \phi\big(\gamma\cdot(\Omega,Z)\big), \tag{7.1}$$

where

$$\Gamma^\infty_{n,m} = \left\{ \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, (\lambda,\mu;\kappa) \right) \in \Gamma_{n,m} \,\Big|\, C = 0 \right\}$$

is a subgroup of $\Gamma_{n,m}$.

We consider the simple case $n=m=1$ and $A=B=1$. A metric $ds^2_{1,1;1,1}$ on $\mathbb{H}_{1,1}$ given by

$$\begin{aligned} ds^2_{1,1;1,1} &= \frac{y+v^2}{y^3}\,(\,dx^2+dy^2\,) + \frac{1}{y}\,(\,du^2+dv^2\,) \\ &\quad - \frac{2v}{y^2}\,(\,dx\,du + dy\,dv\,) \end{aligned}$$

is a G^J-invariant Kähler metric on $\mathbb{H}_{1,1}$. Its Laplacian $\Delta_{1,1;1,1}$ is given by

$$\begin{aligned} \Delta_{1,1;1,1} &= y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) \\ &\quad + (\,y+v^2\,)\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right) \\ &\quad + 2\,y\,v\left(\frac{\partial^2}{\partial x\partial u}+\frac{\partial^2}{\partial y\partial v}\right). \end{aligned}$$

We provide some examples of eigenfunctions of $\Delta_{1,1;1,1}$.

(1) $h(x,y) = y^{\frac12} K_{s-\frac12}(2\pi|a|y)\, e^{2\pi i a x}$ $(s\in\mathbb{C},\ a\neq 0\,)$ with eigenvalue $s(s-1)$. Here

$$K_s(z) := \frac12\int_0^\infty \exp\left\{-\frac{z}{2}(t+t^{-1})\right\}\, t^{s-1}\,dt,$$

where $\mathrm{Re}\, z>0$.

(2) $y^s,\ y^s x,\ y^s u$ $(s\in\mathbb{C})$ with eigenvalue $s(s-1)$.

(3) $y^s v,\ y^s uv,\ y^s xv$ with eigenvalue $s(s+1)$.

(4) $x,\ y,\ u,\ v,\ xv,\ uv$ with eigenvalue 0.

(5) All Maass wave forms.

Let ρ be a rational representation of $GL(n,\mathbb{C})$ on a finite dimensional complex vector space V_ρ. Let $\mathcal{M}\in\mathbb{R}^{(m,m)}$ be a symmetric half-integral semi-positive definite matrix of degree m. Let $C^\infty(\mathbb{H}_{n,m},V_\rho)$ be the algebra of all C^∞ functions on $\mathbb{H}_{n,m}$ with values in V_ρ. We define the $|_{\rho,\mathcal{M}}$-slash action of G^J on $C^\infty(\mathbb{H}_{n,m},V_\rho)$ as follows:

If $f \in C^\infty(\mathbb{H}_{n,m}, V_\rho)$,

$$
\begin{aligned}
& f|_{\rho,\mathcal{M}}[(M,(\lambda,\mu;\kappa))](\Omega,Z) \\
:= \ & e^{-2\pi i\,\mathrm{tr}(\mathcal{M}[Z+\lambda\Omega+\mu](C\Omega+D)^{-1}C)} \cdot e^{2\pi i\,\mathrm{tr}(\mathcal{M}(\lambda\Omega\,{}^t\lambda\,+\,2\lambda\,{}^tZ\,+\,\kappa\,+\,\mu\,{}^t\lambda))} \\
& \times \rho(C\Omega+D)^{-1} f(M\cdot\Omega,(Z+\lambda\Omega+\mu)(C\Omega+D)^{-1}),
\end{aligned} \tag{7.2}
$$

where $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R})$ and $(\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)}$. We recall the Siegel's notation $\alpha[\beta] = {}^t\beta\alpha\beta$ for suitable matrices α and β. We define $\mathbb{D}_{\rho,\mathcal{M}}$ to be the algebra of all differential operators D on $\mathbb{H}_{n,m}$ satisfying the following condition

$$
(Df)|_{\rho,\mathcal{M}}[g] = D(f|_{\rho,\mathcal{M}}[g]) \tag{7.3}
$$

for all $f \in C^\infty(\mathbb{H}_{n,m}, V_\rho)$ and for all $g \in G^J$. We denote by $\mathcal{Z}_{\rho,\mathcal{M}}$ the center of $\mathbb{D}_{\rho,\mathcal{M}}$.

We define an another notion of Maass-Jacobi forms as follows.

Definition 7.2. *A vector-valued smooth function $\phi : \mathbb{H}_{n,m} \longrightarrow V_\rho$ is called a Maass-Jacobi form on $\mathbb{H}_{n,m}$ of type ρ and index $\mathcal{M}$ if it satisfies the following conditions $(MJ1)_{\rho,\mathcal{M}}$, $(MJ2)_{\rho,\mathcal{M}}$ and $(MJ3)_{\rho,\mathcal{M}}$:*

$(MJ1)_{\rho,\mathcal{M}}$ $\phi|_{\rho,\mathcal{M}}[\gamma] = \phi$ *for all* $\gamma \in \Gamma_{n,m}$.

$(MJ2)_{\rho,\mathcal{M}}$ f *is an eigenfunction of all differential operators in the center* $\mathcal{Z}_{\rho,\mathcal{M}}$ *of* $\mathbb{D}_{\rho,\mathcal{M}}$.

$(MJ3)_{\rho,\mathcal{M}}$ f *has a growth condition*

$$
\phi(\Omega,Z) = O\Big(e^{a\det Y} \cdot e^{2\pi tr(\mathcal{M}[V]Y^{-1})}\Big)
$$

as $\det Y \longrightarrow \infty$ *for some* $a > 0$.

The case $n=1,\ m=1$ and $\rho = \det^k (k = 0,1,2,\cdots)$ was studied by R. Bendt and R. Schmidt [1], A. Pitale [16] and K. Bringmann and O. Richter [3]. The case $n=1,\ m=$arbitrary and $\rho = \det^k (k = 1,2,\cdots)$ was dealt with by C. Conley and M. Raum [5]. In [5] the authors proved that the center $\mathcal{Z}_{\det^k,\mathcal{M}}$ of $\mathbb{D}_{\det^k,\mathcal{M}}$ is the polynomial algebra with one generator $\mathcal{C}^{k,\mathcal{M}}$, the so-called *Casimir* operator which is a $|_{\det^k,\mathcal{M}}$-slash invariant differential operator of degree three for the case $n=m=1$ or of degree four for the case $n=1,\ m\geq 2$. Bringmann and Richter [3] considered the Poincaré series $\mathcal{P}_{k,\mathcal{M},s}^{(n,r)}$ (the case $n=m=1$) that is a *harmonic* Maass-Jacobi form in the sense of Definition 7.2 and investigated its Fourier expansion and its Fourier coefficients. Here the *harmonicity* of $\mathcal{P}_{k,\mathcal{M},s}^{(n,r)}$ means that $\mathcal{C}^{k,\mathcal{M}}\mathcal{P}_{k,\mathcal{M},s}^{(n,r)} = 0$, i.e., $\mathcal{P}_{k,\mathcal{M},s}^{(n,r)}$ is an eigenfunction of $\mathcal{C}^{k,\mathcal{M}}$ with zero eigenvalue. Conley and Raum [5] generalized the results in [16] and [3] to the case $n=1$ and m is arbitrary.

Remark 7.2. *In* [2], *Bringmann, Conley and Richter proved that the center of the algebra of differential operators invariant under the action of the Jacobi group over a complex quadratic field is generated by two Casimir operators of degree three. They* *also introduce an analogue of Kohnen's plus space for modular forms of half-integral weight over $K = \mathbb{Q}(i)$, and provide a lift from it to the space of Jacobi forms over K.*

References

[1] R. Berndt and R. Schmidt, *Elements of the Representation Theory of the Jacobi Group*, Progress in Mathematics, **163**, Birkhäuser, Basel, 1998.

[2] K. Bringmann, C. Conley and O. K. Richter, *Jacobi forms over complex quadratic fields via the cubic Casimier operators*, preprint.

[3] K. Bringmann and O. K. Richter, *Zagier-type dualities and lifting maps for harmonic Maass-Jacobi forms*, Advances in Math. **225** (2010), 2298-2315.

[4] D. Bump and Y. J. Choie, *Derivatives of modular forms of negative weight*, Pure Appl. Math. Q. **2** (2006), no. 1, 111-133.

[5] C. Conley and M. Raum, *Harmonic Maass-Jacobi forms of degree 1 with higher rank indices*, arXiv:1012.289/v1 [math.NT] 13 Dec 2010.

[6] M. Eichler and D. Zagier, *The Theory of Jacobi Forms*, Progress in Mathematics, **55**, Birkhäuser, Boston, Basel and Stuttgart, 1985.

[7] Harish-Chandra, *Representations of a semisimple Lie group on a Banach space. I.,* Trans. Amer. Math. Soc. **75** (1953), 185-243.

[8] Harish-Chandra, *The characters of semisimple Lie groups,* Trans. Amer. Math. Soc. **83** (1956), 98-163.

[9] S. Helgason, *Differential operators on homogeneous spaces,* Acta Math. **102** (1959), 239-299.

[10] S. Helgason, *Groups and geometric analysis,* Academic Press, New York (1984).

[11] R. Howe, *Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond,* The Schur lectures (1992) (Tel Aviv), Israel Math. Conf. Proceedings, **vol. 8** (1995), 1-182.

[12] M. Itoh, *On the Yang Problem*, preprint (2011).

[13] H. Maass, *Die Differentialgleichungen in der Theorie der Siegelschen Modulfunktionen*, Math. Ann. **126** (1953), 44–68.

[14] H. Maass, *Siegel modular forms and Dirichlet series,* Lecture Notes in Math., **vol. 216**, Springer-Verlag, Berlin-Heidelberg-New York (1971).

[15] H. Ochiai, *A remark on the generators of invariant differential operators on Siegel-Jacobi space of the smallest size,* preprint (2011).

[16] A. Pitale, *Jacobi Maass forms*, Abh. Math. Sem. Univ. Hamburg **79** (2009), 87-111.

[17] I. Satake, *Algebraic Structures of Symmetric Domains,* Kano Memorial Lectures 4, Iwanami Shoton, Publishers and Princeton University Press (1980).

[18] G. Shimura, *Invariant differential operators on hermitian symmetric spaces,* Ann. Math. **132** (1990), 237-272.

[19] C. L. Siegel, *Symplectic Geometry,* Amer. J. Math. **65** (1943), 1-86; Academic Press, New York and London (1964); Gesammelte Abhandlungen, no. 41, vol. II, Springer-Verlag (1966), 274-359.

[20] C. L. Siegel, *Topics in Complex Function Theory : Abelian Functions and Modular Functions of Several Variables*, vol. III, Wiley-Interscience, 1973.

[21] H. Weyl, *The classical groups: Their invariants and representations,* Princeton Univ. Press, Princeton, New Jersey, second edition (1946).

[22] J.-H. Yang, *The Siegel-Jacobi Operator*, Abh. Math. Sem. Univ. Hamburg **63** (1993), 135–146.

[23] J.-H. Yang, *Singular Jacobi Forms,* Trans. Amer. Math. Soc. **347 (6)** (1995), 2041-2049.

[24] J.-H. Yang, *Construction of vector valued modular forms from Jacobi forms,* Canad. J. Math. **47 (6)** (1995), 1329-1339 *or* arXiv:math.NT/0612502.

[25] J.-H. Yang, *A geometrical theory of Jacobi forms of higher degree,* Proceedings of Symposium on Hodge Theory and Algebraic Geometry (edited by Tadao Oda), Sendai, Japan (1996), 125-147 *or* Kyungpook Math. J. **40 (2)** (2000), 209-237 *or* arXiv:math.NT/0602267.

[26] J.-H. Yang, *The Method of Orbits for Real Lie Groups,* Kyungpook Math. J. **42 (2)** (2002), 199-272 *or* arXiv:math.RT/0602056.

[27] J.-H. Yang, *A note on Maass-Jacobi forms,* Kyungpook Math. J. **43 (4)** (2003), 547–566 *or* arXiv:math.NT/0612387.

[28] J.-H. Yang, *A note on a fundamental domain for Siegel-Jacobi space,* Houston J. Math. **32 (3)** (2006), 701–712 *or* arXiv:math.NT/0507218.

[29] J.-H. Yang, *Invariant metrics and Laplacians on Siegel-Jacobi space,* Journal of Number Theory **127** (2007), 83–102 *or* arXiv:math.NT/0507215.

[30] J.-H. Yang, *A partial Cayley transform for Siegel-Jacobi disk,* J. Korean Math. Soc. **45**, No. 3 (2008), 781-794 *or* arXiv:math.NT/0507216.

[31] J.-H. Yang, *Invariant metrics and Laplacians on Siegel-Jacobi disk,* Chinese Annals of Mathematics, Vol. 31B(1), 2010, 85-100 *or* arXiv:math.NT/0507217.

Department of Mathematics, Inha University, Incheon 402-751, Korea
E-mail address: jhyang@inha.ac.kr