

Problems in the Geometry of the Siegel-Jacobi Space

JAE-HYUN YANG

Department of Mathematics, Inha University, Incheon 22212, Republic of Korea
e-mail: jhyang@inha.ac.kr; jhyang8357@gmail.com

ABSTRACT. The Siegel-Jacobi space is a non-symmetric homogeneous space which is very important geometrically and arithmetically. In this short paper, we propose the basic problems in the geometry of the Siegel-Jacobi space.

1. Introduction

For a given fixed positive integer n , we let

$$\mathbb{H}_n = \{ \Omega \in \mathbb{C}^{(n,n)} \mid \Omega = {}^t\Omega, \operatorname{Im} \Omega > 0 \}$$

be the Siegel upper half plane of degree n and let

$$Sp(n, \mathbb{R}) = \{ M \in \mathbb{R}^{(2n,2n)} \mid {}^tM J_n M = J_n \}$$

be the symplectic group of degree n , where $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F for two positive integers k and l , tM denotes the transposed matrix of a matrix M and

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Then $Sp(n, \mathbb{R})$ acts on \mathbb{H}_n transitively by

$$(1.1) \quad M \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1},$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$ and $\Omega \in \mathbb{H}_n$. Let

$$\Gamma_n = Sp(n, \mathbb{Z}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R}) \mid A, B, C, D \text{ integral} \right\}$$

Received September 5, 2023; revised October 25, 2023; accepted October 26, 2023.

2020 Mathematics Subject Classification: 14G35, 32F45, 32M10, 32Wxx, 53C22.

Key words and phrases: Siegel-Jacobi space, Invariant metrics, Laplace operator, Invariant differential operators, Compactification.

be the Siegel modular group of degree n . This group acts on \mathbb{H}_n properly discontinuously. C. L. Siegel investigated the geometry of \mathbb{H}_n and automorphic forms on \mathbb{H}_n systematically. Siegel [16] found a fundamental domain \mathcal{F}_n for $\Gamma_n \backslash \mathbb{H}_n$ and described it explicitly. Moreover he calculated the volume of \mathcal{F}_n . We also refer to [13, 16] for some details on \mathcal{F}_n .

For two positive integers m and n , we consider the Heisenberg group

$$H_{\mathbb{R}}^{(n,m)} = \{ (\lambda, \mu; \kappa) \mid \lambda, \mu \in \mathbb{R}^{(m,n)}, \kappa \in \mathbb{R}^{(m,m)}, \kappa + \mu^t \lambda \text{ symmetric} \}$$

endowed with the following multiplication law

$$(\lambda, \mu; \kappa) \circ (\lambda', \mu'; \kappa') = (\lambda + \lambda', \mu + \mu'; \kappa + \kappa' + \lambda^t \mu' - \mu^t \lambda')$$

with $(\lambda, \mu; \kappa), (\lambda', \mu'; \kappa') \in H_{\mathbb{R}}^{(n,m)}$. We define the *Jacobi group* G^J of degree n and index m that is the semidirect product of $Sp(n, \mathbb{R})$ and $H_{\mathbb{R}}^{(n,m)}$

$$G^J = Sp(n, \mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)}$$

endowed with the following multiplication law

$$(M, (\lambda, \mu; \kappa)) \cdot (M', (\lambda', \mu'; \kappa')) = (MM', (\tilde{\lambda} + \lambda', \tilde{\mu} + \mu'; \kappa + \kappa' + \tilde{\lambda}^t \mu' - \tilde{\mu}^t \lambda'))$$

with $M, M' \in Sp(n, \mathbb{R}), (\lambda, \mu; \kappa), (\lambda', \mu'; \kappa') \in H_{\mathbb{R}}^{(n,m)}$ and $(\tilde{\lambda}, \tilde{\mu}) = (\lambda, \mu)M'$. Then G^J acts on $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$ transitively by

$$(1.2) \quad (M, (\lambda, \mu; \kappa)) \cdot (\Omega, Z) = \left(M \cdot \Omega, (Z + \lambda\Omega + \mu)(C\Omega + D)^{-1} \right),$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R}), (\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(n,m)}$ and $(\Omega, Z) \in \mathbb{H}_n \times \mathbb{C}^{(m,n)}$. We note that the Jacobi group G^J is *not* a reductive Lie group and the homogeneous space $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$ is not a symmetric space. From now on, for brevity we write $\mathbb{H}_{n,m} = \mathbb{H}_n \times \mathbb{C}^{(m,n)}$. The homogeneous space $\mathbb{H}_{n,m}$ is called the *Siegel-Jacobi space* of degree n and index m .

In this short article, we propose the basic and natural problems in the geometry of the Siegel-Jacobi space.

Notations: We denote by \mathbb{Q}, \mathbb{R} and \mathbb{C} the field of rational numbers, the field of real numbers and the field of complex numbers respectively. We denote by \mathbb{Z} the ring of integers. The symbol “:=” means that the expression on the right is the definition of that on the left. For two positive integers k and l , $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F . For a square matrix $A \in F^{(k,k)}$ of degree k , $\sigma(A)$ denotes the trace of A . For any $M \in F^{(k,l)}$, ${}^t M$ denotes the transpose of a matrix M . I_n denotes the identity matrix of degree n .

For a complex matrix A , \bar{A} denotes the complex *conjugate* of A . For a number field F , we denote by \mathbb{A}_F the ring of adèles of F . If $F = \mathbb{Q}$, the subscript will be omitted.

2. Brief Review on the Geometry of the Siegel Space

We let $G := Sp(n, \mathbb{R})$ and $K = U(n)$. The stabilizer of the action (1.1) at iI_n is

$$\left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mid A + iB \in U(n) \right\} \cong U(n).$$

Thus we get the biholomorphic map

$$G/K \longrightarrow \mathbb{H}_n, \quad gK \mapsto g \cdot iI_n, \quad g \in G.$$

\mathbb{H}_n is a Hermitian symmetric manifold.

For $\Omega = (\omega_{ij}) \in \mathbb{H}_n$, we write $\Omega = X + iY$ with $X = (x_{ij})$, $Y = (y_{ij})$ real. We put $d\Omega = (d\omega_{ij})$ and $d\bar{\Omega} = (d\bar{\omega}_{ij})$. We also put

$$\frac{\partial}{\partial \Omega} = \left(\frac{1 + \delta_{ij}}{2} \frac{\partial}{\partial \omega_{ij}} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{\Omega}} = \left(\frac{1 + \delta_{ij}}{2} \frac{\partial}{\partial \bar{\omega}_{ij}} \right).$$

C. L. Siegel [16] introduced the symplectic metric $ds_{n;A}^2$ on \mathbb{H}_n invariant under the action (1.1) of $Sp(n, \mathbb{R})$ that is given by

$$(2.1) \quad ds_{n;A}^2 = A \sigma(Y^{-1} d\Omega Y^{-1} d\bar{\Omega}), \quad A > 0.$$

It is known that the metric $ds_{n;A}^2$ is a Kähler-Einstein metric. H. Maass [12] proved that its Laplace operator $\Delta_{n;A}$ is given by

$$(2.2) \quad \Delta_{n;A} = \frac{4}{A} \sigma \left(Y^t \left(Y \frac{\partial}{\partial \bar{\Omega}} \right) \frac{\partial}{\partial \Omega} \right).$$

And

$$(2.3) \quad dv_n(\Omega) = (\det Y)^{-(n+1)} \prod_{1 \leq i \leq j \leq n} dx_{ij} \prod_{1 \leq i \leq j \leq n} dy_{ij}$$

is a $Sp(n, \mathbb{R})$ -invariant volume element on \mathbb{H}_n (cf. [17, p. 130]).

Siegel proved the following theorem for the Siegel space $(\mathbb{H}_n, ds_{n;1}^2)$.

Theorem 2.1. (Siegel [16]). (1) *There exists exactly one geodesic joining two arbitrary points Ω_0, Ω_1 in \mathbb{H}_n . Let $R(\Omega_0, \Omega_1)$ be the cross-ratio defined by*

$$R(\Omega_0, \Omega_1) = (\Omega_0 - \Omega_1)(\Omega_0 - \bar{\Omega}_1)^{-1}(\bar{\Omega}_0 - \bar{\Omega}_1)(\bar{\Omega}_0 - \Omega_1)^{-1}.$$

For brevity, we put $R_* = R(\Omega_0, \Omega_1)$. Then the symplectic length $\rho(\Omega_0, \Omega_1)$ of the geodesic joining Ω_0 and Ω_1 is given by

$$\rho(\Omega_0, \Omega_1)^2 = \sigma \left(\left(\log \frac{1 + R_*^{\frac{1}{2}}}{1 - R_*^{\frac{1}{2}}} \right)^2 \right),$$

where

$$\left(\log \frac{1 + R_*^{\frac{1}{2}}}{1 - R_*^{\frac{1}{2}}} \right)^2 = 4R_* \left(\sum_{k=0}^{\infty} \frac{R_*^k}{2k+1} \right)^2.$$

(2) For $M \in Sp(n, \mathbb{R})$, we set

$$\tilde{\Omega}_0 = M \cdot \Omega_0 \quad \text{and} \quad \tilde{\Omega}_1 = M \cdot \Omega_1.$$

Then $R(\Omega_1, \Omega_0)$ and $R(\tilde{\Omega}_1, \tilde{\Omega}_0)$ have the same eigenvalues.

(3) All geodesics are symplectic images of the special geodesics

$$\alpha(t) = i \operatorname{diag}(a_1^t, a_2^t, \dots, a_n^t),$$

where a_1, a_2, \dots, a_n are arbitrary positive real numbers satisfying the condition

$$\sum_{k=1}^n (\log a_k)^2 = 1.$$

The proof of the above theorem can be found in [16, pp. 289-293].

Let $\mathbb{D}(\mathbb{H}_n)$ be the algebra of all differential operators on \mathbb{H}_n invariant under the action (1.1). Then according to Harish-Chandra [5, 6],

$$\mathbb{D}(\mathbb{H}_n) = \mathbb{C}[D_1, \dots, D_n],$$

where D_1, \dots, D_n are algebraically independent invariant differential operators on \mathbb{H}_n . That is, $\mathbb{D}(\mathbb{H}_n)$ is a commutative algebra that is finitely generated by n algebraically independent invariant differential operators on \mathbb{H}_n . Maass [13] found the explicit D_1, \dots, D_n . Let $\mathfrak{g}_{\mathbb{C}}$ be the complexification of the Lie algebra of G . It is known that $\mathbb{D}(\mathbb{H}_n)$ is isomorphic to the center of the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$ (cf. [7]).

Example 2.2. We consider the simplest case $n = 1$ and $A = 1$. Let \mathbb{H} be the Poincaré upper half plane. Let $\omega = x + iy \in \mathbb{H}$ with $x, y \in \mathbb{R}$ and $y > 0$. Then the Poincaré metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2} = \frac{d\omega d\bar{\omega}}{y^2}$$

is a $SL(2, \mathbb{R})$ -invariant Kähler-Einstein metric on \mathbb{H} . The geodesics of (\mathbb{H}, ds^2) are either straight vertical lines perpendicular to the x -axis or circular arcs perpendicular to the x -axis (half-circles whose origin is on the x -axis). The Laplace operator Δ of (\mathbb{H}, ds^2) is given by

$$\Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

and

$$dv = \frac{dx \wedge dy}{y^2}$$

is a $SL(2, \mathbb{R})$ -invariant volume element. The scalar curvature, i.e., the Gaussian curvature is -1 . The algebra $\mathbb{D}(\mathbb{H})$ of all $SL(2, \mathbb{R})$ -invariant differential operators on \mathbb{H} is given by

$$\mathbb{D}(\mathbb{H}) = \mathbb{C}[\Delta].$$

The distance between two points $\omega_1 = x_1 + iy_1$ and $\omega_2 = x_2 + iy_2$ in (\mathbb{H}, ds^2) is given by

$$\begin{aligned} \rho(\omega_1, \omega_2) &= 2 \ln \frac{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} + \sqrt{(x_2 - x_1)^2 + (y_2 + y_1)^2}}{2\sqrt{y_1 y_2}} \\ &= \cosh^{-1} \left(1 + \frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{2y_1 y_2} \right) \\ &= 2 \sinh^{-1} \frac{1}{2} \sqrt{\frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{y_1 y_2}}. \end{aligned}$$

3. Basic Problems in the Geometry of the Siegel-Jacobi Space

For a coordinate $(\Omega, Z) \in \mathbb{H}_{n,m}$ with $\Omega = (\omega_{\mu\nu})$ and $Z = (z_{kl})$, we put $d\Omega, d\bar{\Omega}, \frac{\partial}{\partial\Omega}, \frac{\partial}{\partial\bar{\Omega}}$ as before and set

$$\begin{aligned} Z &= U + iV, & U &= (u_{kl}), & V &= (v_{kl}) \text{ real,} \\ dZ &= (dz_{kl}), & d\bar{Z} &= (d\bar{z}_{kl}), \end{aligned}$$

$$\frac{\partial}{\partial Z} = \begin{pmatrix} \frac{\partial}{\partial z_{11}} & \cdots & \frac{\partial}{\partial z_{m1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial z_{1n}} & \cdots & \frac{\partial}{\partial z_{mn}} \end{pmatrix}, \quad \frac{\partial}{\partial \bar{Z}} = \begin{pmatrix} \frac{\partial}{\partial \bar{z}_{11}} & \cdots & \frac{\partial}{\partial \bar{z}_{m1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial \bar{z}_{1n}} & \cdots & \frac{\partial}{\partial \bar{z}_{mn}} \end{pmatrix}.$$

The author proved the following theorems in [18].

Theorem 3.1. For any two positive real numbers A and B ,

$$\begin{aligned} ds_{n,m;A,B}^2 &= A \sigma \left(Y^{-1} d\Omega Y^{-1} d\bar{\Omega} \right) \\ &\quad + B \left\{ \sigma \left(Y^{-1} {}^t V V Y^{-1} d\Omega Y^{-1} d\bar{\Omega} \right) + \sigma \left(Y^{-1} {}^t (dZ) d\bar{Z} \right) \right. \\ &\quad \left. - \sigma \left(V Y^{-1} d\Omega Y^{-1} {}^t (d\bar{Z}) \right) - \sigma \left(V Y^{-1} d\bar{\Omega} Y^{-1} {}^t (dZ) \right) \right\} \end{aligned}$$

is a Riemannian metric on $\mathbb{H}_{n,m}$ which is invariant under the action (1.2) of G^J .

Proof. See [18, Theorem 1.1]. \square

Theorem 3.2. The Laplace operator $\Delta_{m,m;A,B}$ of the G^J -invariant metric $ds_{n,m;A,B}^2$ is given by

$$(3.1) \quad \Delta_{n,m;A,B} = \frac{4}{A} \mathbb{M}_1 + \frac{4}{B} \mathbb{M}_2,$$

where

$$\begin{aligned} \mathbb{M}_1 &= \sigma \left(Y {}^t \left(Y \frac{\partial}{\partial \bar{\Omega}} \right) \frac{\partial}{\partial \Omega} \right) + \sigma \left(V Y^{-1} {}^t V \left(Y \frac{\partial}{\partial \bar{Z}} \right) \frac{\partial}{\partial Z} \right) \\ &\quad + \sigma \left(V {}^t \left(Y \frac{\partial}{\partial \bar{\Omega}} \right) \frac{\partial}{\partial Z} \right) + \sigma \left({}^t V \left(Y \frac{\partial}{\partial \bar{Z}} \right) \frac{\partial}{\partial \Omega} \right) \end{aligned}$$

and

$$\mathbb{M}_2 = \sigma \left(Y \frac{\partial}{\partial Z} \left(\frac{\partial}{\partial \bar{Z}} \right) \right).$$

Furthermore \mathbb{M}_1 and \mathbb{M}_2 are differential operators on $\mathbb{H}_{n,m}$ invariant under the action (1.2) of G^J .

Proof. See [18, Theorem 1.2]. \square

Remark 3.3. Erik Balslev [2] developed the spectral theory of $\Delta_{1,1;1,1}$ on $H_{1,1}$ for certain arithmetic subgroups of the Jacobi modular group to prove that the set of all eigenvalues of $\Delta_{1,1;1,1}$ satisfies the Weyl law.

Remark 3.4. The scalar curvature of $(\mathbb{H}_{1,1}, ds_{1,1;A,B}^2)$ is $-\frac{3}{A}$ and hence is independent of the parameter B . We refer to [21] for more detail.

Remark 3.5. Yang and Yin [22] showed that $ds_{n,m;A,B}^2$ is a Kähler metric. For some applications of the invariant metric $ds_{n,m;A,B}^2$ we refer to [22].

Now we propose the basic and natural problems.

Problem 1. Find all the geodesics of $(\mathbb{H}_{n,m}, ds_{n,m;A,B}^2)$ explicitly.

Problem 2. Compute the distance between two points (Ω_1, Z_1) and (Ω_2, Z_2) of $\mathbb{H}_{n,m}$ explicitly.

Problem 3. Compute the Ricci curvature tensor and the scalar curvature of $(\mathbb{H}_{n,m}, ds_{n,m;A,B}^2)$.

Problem 4. Find all the eigenfunctions of the Laplace operator $\Delta_{n,m;A,B}$.

Problem 5. Develop the spectral theory of $\Delta_{n,m;A,B}$.

Problem 6. Describe the algebra of all G^J -invariant differential operators on $\mathbb{H}_{n,m}$ explicitly. We refer to [19, 20, 22] for some details.

Problem 7. Find the trace formula for the Jacobi group $G^J(\mathbb{A})$.

Problem 8. Discuss the behaviour of the analytic torsion of the Siegel-Jacobi space $\mathbb{H}_{n,m}$ or the arithmetic quotients of $\mathbb{H}_{n,m}$.

We make some remarks on the above problems.

Remark 3.6. Problem 1 reduces to trying to solve a system of ordinary differential equations explicitly. If Problem 2 is solved, the distance formula would be a very beautiful one that generalizes the distance formula $\rho(\Omega_0, \Omega_1)$ given by Theorem 2.1 (the Siegel space case).

Remark 3.7. Problem 3 was recently solved in the case that $n = 1$ and m is arbitrary. Precisely the scalar and Ricci curvatures of the Siegel-Jacobi space $(\mathbb{H}_{1,m}, ds_{1,m;A,B}^2)$ ($m \geq 1$) were completely computed by G. Khan and J. Zhang [8, Proposition 8, pp. 825–826]. Furthermore Khan and Zhang proved that $(\mathbb{H}_{1,m}, ds_{1,m;A,B}^2)$ ($m \geq 1$) has non-negative orthogonal anti-bisectional curvature (cf. [8, Proposition 9, p. 826]).

Remark 3.8. Concerning Problem 4 and Problem 5, computing eigenfunctions explicitly is a tall order, but if this can be done it will shed a lot of light onto the geometry of this space. And understanding the spectral geometry seems to be a central question which will likely have applications in number theory and other areas.

Remark 3.9. The algebra $\mathbb{D}(\mathbb{H}_{n,m})$ of all G^J -invariant differential operators on $\mathbb{H}_{n,m}$ is not commutative. Concerning Problem 6, the case $n = m = 1$ was completely solved by M. Itoh, H. Ochiai and J.-H. Yang in 2013. They proved that the noncommutative algebra $\mathbb{D}(\mathbb{H}_{1,1})$ is generated by four explicit generators D_1, D_2, D_3, D_4 , and found the relations among those D_i ($1 \leq i \leq 4$). For more precise statements, we refer to [19, pp. 56–58] and [20, pp. 285–290]. We note that the above four generators D_i ($1 \leq i \leq 4$) are not algebraically independent.

Remark 3.10. The solution of Problem 7 will provide lots of arithmetic properties of the Siegel-Jacobi space.

4. Final Remarks

Let $\Gamma_n(N)$ be the principal congruence subgroup of the Siegel modular group Γ_n . Let $\mathfrak{X}_n(N) := \Gamma_n(N) \backslash \mathbb{H}_n$ be the moduli of n -dimensional principally polarized abelian varieties with level N -structure. The Mumford school [1] found toroidal compactifications of $\mathfrak{X}_n(N)$ which are usefully applied in the study of the geometry and arithmetic of $\mathfrak{X}_n(N)$. D. Mumford [14] proved the Hirzebruch's Proportionality Theorem in the non-compact case introducing a *good singular* Hermitian metric on an automorphic vector bundle on a smooth toroidal compactification of $\mathfrak{X}_n(N)$ with $N \geq 3$.

We set

$$\Gamma_{n,m}(N) := \Gamma_n \times H_{\mathbb{Z}}^{(n,m)},$$

where

$$H_{\mathbb{Z}}^{(n,m)} = \left\{ (\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(n,m)} \mid \lambda, \mu, \kappa \text{ integral} \right\}.$$

Let

$$\mathfrak{X}_{n,m}(N) := \Gamma_{n,m}(N) \backslash \mathbb{H}_{n,m}$$

be the universal abelian variety. An arithmetic toroidal compactification of $\mathfrak{X}_{n,m}(N)$ was intensively investigated by R. Pink [15]. D. Mumford described very nicely a toroidal compactification of the universal elliptic curve $\mathfrak{X}_{1,1}(N)$ (cf. [1, pp. 14–25]). The geometry of $\mathfrak{X}_{n,m}(N)$ is closely related to the theory of Jacobi forms (cf. [3, 9, 10, 11]). Jacobi forms play an important role in the study of the geometric and arithmetic of $\mathfrak{X}_{n,m}(N)$. We refer to [4, 23] for the theory of Jacobi forms.

References

- [1] A. Ash, D. Mumford, M. Rapoport and Y.-S. Tai, *Smooth Compactifications of Locally Symmetric Varieties*, Cambridge Math. Lib., Cambridge University Press, Cambridge, 2010, x+230 pp.
- [2] E. Balslev, *Spectral theory of the Laplacian on the modular Jacobi group manifold*, preprint, Aarhus University (2012).
- [3] J. I. Burgos Gil, J. Kramer and U. Kühn, *The Singularities of the invariant metric on the Jacobi line bundle*, London Math. Soc. Lecture Note Ser. **427**, Cambridge University Press, Cambridge, 2016, 45–77.
- [4] M. Eichler and D. Zagier, *The Theory of Jacobi Forms*, Progr. Math. **55**, Birkhäuser Boston, Inc., Boston, MA, 1985, v+148 pp.
- [5] Harish-Chandra, *Representations of a semisimple Lie group on a Banach space. I*, Trans. Amer. Math. Soc., **75**(1953), 185–243.

- [6] Harish-Chandra, *The characters of semisimple Lie groups*, Trans. Amer. Math. Soc., **83**(1956), 98–163.
- [7] S. Helgason, *Groups and geometric analysis*, Pure Appl. Math. **113**, Academic Press, Inc., Orlando, FL, 1984, xix+654 pp.
- [8] G. Khan and J. Zhang, *A hall of statistical mirrors*, Asian J. Math., **26**(6)(2022), 809–846.
- [9] J. Kramer, *A geometrical approach to the theory of Jacobi forms*, Compositio Math., **79**(1)(1991), 1–19.
- [10] J. Kramer, *A geometrical approach to Jacobi forms, revisited*, 2014, Oberwolfach reports from the workshop on modular forms held April 22–May 2, 2014, organized by J. H. Bruinier, A. Ichino, T. Ikeda and Ö. Imamoglu, Report No. 22/2014. DOI: 10.4171/OWR/2014/22.
- [11] J. Kramer and A.-M. von Pippich, *Snapshots of modern mathematics from Oberwolfach: Special Values of Zeta Functions and Areas of Triangles*, Notices Amer. Math. Soc., **63**(8)(2016), 917–922.
- [12] H. Maass, *Die Differentialgleichungen in der Theorie der Siegelschen Modulfunktionen*, Math. Ann., **126**(1953), 44–68.
- [13] H. Maass, *Siegel's modular forms and Dirichlet series*, Lecture Notes in Math. **216**, Springer-Verlag, Berlin-New York, 1971, v+328 pp.
- [14] D. B. Mumford, *Hirzebruch's Proportionality Theorem in the Non-Compact Case*, Invent. Math., **42**(1977), 239–272.
- [15] R. Pink, *Arithmetical compactification of mixed Shimura varieties*, Bonner Math. Schriften, **209** [Bonn Mathematical Publications] Universität Bonn, Mathematisches Institut, Bonn, 1990, xviii+340 pp.
- [16] C. L. Siegel, *Symplectic Geometry*, Amer. J. Math. **65**(1943), 1–86; Academic Press, New York and London (1964); Gesammelte Abhandlungen, no. 41, vol. II, Springer-Verlag (1966), 274–359.
- [17] C. L. Siegel, *Topics in complex function theory. Vol. III: Abelian Functions and Modular Functions of Several Variables*, Wiley Classics Lib. Wiley-Intersci. Publ., John Wiley & Sons, Inc., New York, 1989, x+244 pp.
- [18] J.-H. Yang, *Invariant metrics and Laplacians on Siegel-Jacobi space*, J. Number Theory, **127**(1)(2007), 83–102.
- [19] J.-H. Yang, *Invariant differential operators on Siegel-Jacobi space and Maass-Jacobi forms*, Proceedings of the International Conference (October 10–12, 2012) on Geometry, number theory and representation theory, 37–63. KM Kyung Moon Sa, Seoul (2013). ISBN:978-89-88615-35-5.
- [20] J.-H. Yang, *Geometry and Arithmetic on the Siegel-Jacobi Space*, Geometry and Analysis on Manifolds, In Memory of Professor Shoshichi Kobayashi (edited by T. Ochiai, A. Weinstein et al), Progr. Math. **308**, Birkhäuser, Springer International Publishing AG Switzerland (2015), 275–325.
- [21] J.-H. Yang, Y.-H. Yong, S.-N. Huh, J.-H. Shin and G.-H. Min, *Sectional Curvatures of the Siegel-Jacobi Space*, Bull. Korean Math. Soc., **50**(3)(2013), 787–799.

- [22] J. Yang and L. Yin, *Differential Operators for Siegel-Jacobi forms*, *Sci. China Math.*, **59(6)**(2016), 1029–1050.
- [23] C. Ziegler, *Jacobi Forms of Higher Degree*, *Abh. Math. Sem. Univ. Hamburg*, **59**(1989), 191–224.