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Langlands Functoriality Conjecture

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To the memory of my parents

Abstract. Functoriality conjecture is one of the central and influential subjects of the present day mathematics. Functoriality is the profound lifting problem formulated by Robert Langlands in the late 1960s in order to establish nonabelian class field theory. In this expository article, I describe the Langlands-Shahidi method, the local and global Langlands conjectures and the converse theorems which are powerful tools for the establishment of functoriality of some important cases, and survey the interesting results related to functoriality conjecture.

1. Introduction

Functoriality Conjecture or the Principle of Functoriality is the profound question that was raised and formulated by Robert P. Langlands in the late 1960s to establish nonabelian class field theory and its reciprocity law. Functoriality conjecture describes deep relationships among automorphic representations on different groups. This conjecture can be described in a rough form as follows: To every L-homomorphism $\varphi : {}^LH \longrightarrow {}^LG$ between the L-groups of H and G that are quasi-split reductive groups, there exists a *natural* lifting or transfer of automorphic representations of H to those of G. In 1978, Gelbart and Jacquet [22] established an example of the functoriality for the symmetric square Sym^2 of $GL(2)$ using the converse theorem on $GL(3)$. In 2002 after 24 years, Kim and Shahidi [46] established the functoriality for the symmetric cube Sym^3 of $GL(2)$ and thereafter Kim [43] proved the validity of the functoriality for the symmetric fourth Sym^4 of $GL(2)$. These results have led to breakthroughs toward certain important conjectures in number theory, those of Ramanujan, Selberg and Sato-Tate conjectures. We refer to [81] for more detail on applications to the progress made toward to the conjectures just mentioned. Recently the Sato-Tate conjecture for an important class of cases related to elliptic curves has been verified by Clozel, Harris, Shepherd-Barron

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and Taylor [11], [27], [69], [83]. Past ten years the functoriality for the tensor product $GL(2) \times GL(2) \longrightarrow GL(4)$ by Ramakrishnan [71], for the tensor product $GL(2) \times GL(3) \longrightarrow GL(6)$ by Kim and Shahidi [46], for the exterior square of $GL(4)$ by Kim [43] and the weak functoriality to $GL(N)$ for generic cuspidal representations of split classical groups $SO(2n+1)$, $Sp(2n)$ and $SO(2n)$ by Cogdell, Kim, Piatetski-Shapiro and Shahidi [13], [14] were established by applying appropriate converse theorems of Cogdell and Piatetski-Shapiro [15], [16] to analytical properties of certain automorphic L-functions arising from the Langlands-Shahidi method. In fact, the Langlands functoriality was established only for very special L-homomorphisms between the L-groups. It is natural to ask how to find a larger class of certain L-homomorphisms for which the functoriality is valid. It is still very difficult to answer this question.

The Arthur-Selberg trace formula has also provided some instances of Langlands functoriality (see [4], [6], [61], [62]). In a certain sense, it seems that the trace formula is a useful and powerful tool to tackle the functoriality conjecture. Nevertheless the incredible power of Langlands functoriality seems beyond present technology and knowledge. We refer the reader to [66] for Langlands' comments on the limitations of the trace formula. Special cases of functoriality arises naturally from the conjectural theory of endoscopy (cf. [50]), in which a comparison of trace formulas would be used to characterize the internal structure of automorphic representations of a given group. I shall not deal with the trace formula, the base change and the theory of endoscopy in this article. Nowadays local and global Langlands conjectures are believed to be encompassed in the functoriality (cf. [65], [66]). Quite recently Khare, Larsen and Savin [41], [42] made a use of the functorial lifting from $SO(2n+1)$ to $GL(2n)$, from $Sp(2n)$ to $GL(2n+1)$ and the theta lifting of the exceptional group G_2 to $Sp(6)$ to prove that certain finite simple groups $PSp_n(\mathbb{F}_{\ell^k})$, $G_2(\mathbb{F}_{\ell^k})$ and $SO_{2n+1}(\mathbb{F}_{\ell^k})$ with some mild restrictions appear as Galois groups over $\mathbb{Q}$.

This paper is organized as follows. In Section 2, we review the notion of automorphic L-functions and survey the Langlands-Shahidi method briefly following closely the article of Shahidi [77]. I would like to recommend to the reader two lecture notes which were very nicely written by Cogdell [12] and Kim [44] for more information on automorphic L-functions. In Section 3, we review the Weil-Deligne group briefly and formulate the local Langlands conjecture. We describe the recent results about the local Langlands conjecture for $GL(n)$ and $SO(2n+1)$. In Section 4, we discuss the global Langlands conjecture which is still not well formulated in the number field case. The work on the global Langlands conjecture for $GL(2)$ over a function field done by Drinfeld was extended by Lafforgue ten years ago to give a proof of the global Langlands conjecture for $GL(n)$ over a function field. We will not deal with the function field case in this article. We refer to [52] for more detail. Unfortunately there is very little known of the global Langlands conjecture in the number field case. I have an audacity to mention the Langlands hypothetical group and the hypothetical motivic Galois group following the line of Arthur's argument in [3]. In Section 5, I formulate the Langlands functoriality conjecture in several

ways and describe the striking examples of Langlands functoriality established past ten years. I want to mention that there is a descent method of studying the opposite direction of the lift initiated by Ginzburg, Rallis and Soudry (see [25], [39]). I shall not deal with the descent method here. In the appendix, I describe a brief history of the converse theorems obtained by Hamburger, Hecke, Weil, Cogdell, Piatetski-Shapiro, Jinag and Soudry. I present the more or less exact formulations of the converse theorems. As mentioned earlier, the converse theorems play a crucial role in establishing the functoriality for the examples discussed in Section 5.

Notations: We denote by $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the field of rational numbers, the field of real numbers and the field of complex numbers respectively. We denote by $\mathbb{R}_+^*$ the multiplicative group of positive real numbers. $\mathbb{C}^*$ (resp. $\mathbb{R}^*$) denotes the multiplicative group of nonzero complex(resp. real) numbers. We denote by $\mathbb{Z}$ and $\mathbb{Z}^+$ the ring of integers and the set of all positive integers respectively. For a number field F, we denote by $\mathbb{A}_F$ and $\mathbb{A}_F^*$ the ring of adeles of F and the multiplicative group of ideles of F respectively. If there is no confusion, we write simply $\mathbb{A}$ and $\mathbb{A}^*$ instead of $\mathbb{A}_F$ and $\mathbb{A}_F^*$. For a field k we denote by Γ_k the Galois group $Gal(\overline{k}/k)$, where $\overline{k}$ is a separable algebraic closure of k. We denote by $\mathbb{G}_m$ the multiplicative group in one variable. $\mathbb{G}_a$ denotes the additive group in one variable.

2. Automorphic L-functions

Let G be a connected, quasi-split reductive group over a number field F. For each place v of F, we let F_v be the completion of F, $\mathfrak{o}_v$ the rings of integers of F_v, $\mathfrak{p}_v$ the maximal ideal of $\mathfrak{o}_v$, and let q_v be the order of the residue field $k_v = \mathfrak{o}_v/\mathfrak{p}_v$. We denote by $\mathbb{A} = \mathbb{A}_F$ the ring of adeles of F. We fix a Borel subgroup B of G over F. Write $B = TU$, where T is a maximal torus and U is the unipotent radical, both over F. Let P be a parabolic subgroup of G. Assume $P \supset B$. Let $P = MN$ be a Levi decomposition of P with Levi factor M and its unipotent radical N. Then $N \subset U$. For each place v of F, we let $G_v = G(F_v)$. Similarly we use B_v, T_v, U_v, P_v, M_v and N_v to denote the corresponding groups of F_v-rational points. Let $G(\mathbb{A}), B(\mathbb{A}), \cdots, N(\mathbb{A})$ be the corresponding adelic groups for the subgroups defined before. When G is unramified over a place v in the sense that G is quasi-split over F_v and that G is split over a finite unramified extension of F_v, we let $K_v = G(\mathfrak{o}_v)$. Otherwise we shall fix a special maximal compact subgroup $K_v \subset G_v$. We set $K_{\mathbb{A}} = \otimes_v K_v$. Then $G(\mathbb{A}) = P(\mathbb{A})K_{\mathbb{A}}$.

Let $\Pi = \otimes_v \Pi_v$ be a cuspidal automorphic representation of $G(\mathbb{A})$. We refer to [36], [37], [59] for the notion of automorphic representations. Let S be a finite set of places including all archimedean ones such that both Π_v and G_v are unramified for any place $v \notin S$. Then for each $v \notin S$, Π_v determines uniquely a semi-simple conjugacy class $c(\Pi_v)$ in the L-group ${}^L G_v$ of G_v as a group defined over F_v. We refer to [9], [48] for the definition and construction of the L-group. We note that there exists a natural homomorphism $\xi_v : {}^L G_v \longrightarrow {}^L G$. For a finite dimensional representation r of ${}^L G$, putting $r_v = r \circ \xi_v$, the local Langlands L-function $L(s, \Pi_v, r_v)$

associated to Π_v and r_v is defined to be (cf. [9], [54])

$$L(s,\Pi_v,r_v)=\det\Big(I-r_v\big(c(\Pi_v)\big)q_v^{-s}\Big)^{-1}. \tag{2.1}$$

We set

$$L_S(s,\Pi,r)=\prod_{v\notin S}L(s,\Pi_v,r_v). \tag{2.2}$$

Langlands (cf. [54]) proved that $L_S(s,\Pi,r)$ converges absolutely for sufficiently large $\mathrm{Re}(s)>0$ and defines a holomorphic function there. Furthermore he proposed the following question.

Conjecture A(Langlands, [54])**.** $L_S(s,\Pi,r)$ has a meromorphic continuation to the whole complex plane and satisfies a standard functional equation.

F. Shahidi (cf. [77], [78]) gave a partial answer to the above conjecture using the so-called Langlands-Shahidi method. I shall describe Shahidi's results briefly following his article [77].

Let P be a maximal parabolic subgroup of G and $P=MN$ its Levi decomposition with its Levi factor M and its unipotent radical N. Let A be the split torus in the center of M. For every group H defined over F, we let $X(H)_F$ be the group of F-rational characters of H. We set

$$\mathfrak{a}=\mathrm{Hom}\big(X(M)_F,\mathbb{R}\big).$$

Then

$$\mathfrak{a}^*=X(M)_F\otimes_{\mathbb{Z}}\mathbb{R}\cong X(A)_F\otimes_{\mathbb{Z}}\mathbb{R}.$$

We set $\mathfrak{a}_{\mathbb{C}}^*:=\mathfrak{a}^*\otimes_{\mathbb{R}}\mathbb{C}$. Let $\mathfrak{z}$ be the real Lie algebra of the split torus in the center $C(G)$ of G. Then $\mathfrak{z}\subset\mathfrak{a}$ and $\mathfrak{a}/\mathfrak{z}$ is of dimension 1. The imbedding $X(M)_F\hookrightarrow X(M)_{F_v}$ induces an imbedding $\mathfrak{a}_v\hookrightarrow\mathfrak{a}$, where $\mathfrak{a}_v=\mathrm{Hom}\big(X(M)_{F_v},\mathbb{R}\big)$. The Harish-Chandra homomorphism $H_P:M(\mathbb{A})\longrightarrow\mathfrak{a}$ is defined by

$$\exp\langle\chi,H_P(m)\rangle=\prod_v|\chi(m_v)|_v,\quad \chi\in X(M)_F,\ m=\otimes_v m_v\in M(\mathbb{A}).$$

We may extend H_P to $G(\mathbb{A})$ by letting it trivial on $N(\mathbb{A})$ and $K_{\mathbb{A}}$. We define $H_{P_v}:M_v\longrightarrow\mathfrak{a}_v$ by

$$q_v^{\langle\chi_v,H_{P_v}(m)\rangle}=|\chi_v(m_v)|_v,\quad \chi_v\in X(M)_{F_v},\ m_v\in M_v$$

for a finite place v, and define

$$\exp\langle\chi_v,H_{P_v}(m)\rangle=|\chi_v(m_v)|_v,\quad \chi_v\in X(M)_{F_v},\ m_v\in M_v$$

for an infinite place v. Then we have

$$\exp\langle \chi, H_P(m)\rangle = \prod_{v=\infty} \exp\langle \chi, H_{P_v}(m)\rangle \cdot \prod_{v<\infty} q_v^{\langle \chi, H_{P_v}(m)\rangle}, \tag{2.3}$$

where $\chi \in X(M)_F$ and $m = \otimes_v m_v \in M(\mathbb{A})$. We observe that the product in (2.3) is finite.

Let A_0 be the maximal F-split torus in T. We denote by Φ the set of roots of A_0. Then $\Phi = \Phi^+ \cup \Phi^-$, where Φ^+ is the set of roots generating U and $\Phi^- = -\Phi^+$. Let $\Delta \subset \Phi^+$ be the set of simple roots. The unique reduced root of A in N can be identified by an element $\alpha \in \Delta$. Let ρ_P be half the sum of roots generating N. We set

$$\widetilde{\alpha} = \langle \rho_P, \alpha\rangle^{-1} \rho_P.$$

Here, for any pair of roots α and β in Φ^+, the pairing $\langle \alpha, \beta\rangle$ is defined as follows. Let $\widetilde{\Phi}^+$ be the set of non-restricted roots of T in U. We see that the set of simple roots $\widetilde{\Delta}$ in $\widetilde{\Phi}^+$ restricts to Δ. Identifying α and β with roots in $\widetilde{\Phi}^+$, we set

$$\langle \alpha, \beta\rangle = \frac{2(\alpha, \beta)}{(\beta, \beta)},$$

where $(\ ,\ )$ is the standard inner product on $\mathbb{R}^l$ with $l = |\widetilde{\Delta}|$.

Let $\pi = \otimes_v \pi_v$ be a cuspidal automorphic representation of $M(\mathbb{A})$. Given a $K_{\mathbb{A}} \cap M(\mathbb{A})$-finite function ϕ in the representation space V_π of π, we may extend ϕ to a function $\widetilde{\phi}$ on $G(\mathbb{A})$ (cf. [75]). Then the Eisenstein series $E(s, \widetilde{\phi}, g, P)$ is defined by

$$E(s, \widetilde{\phi}, g, P) = \sum_{\gamma \in P(F)\backslash G(F)} \widetilde{\phi}(\gamma g) \exp\langle s\widetilde{\alpha} + \rho_P, H_P(\gamma g)\rangle, \quad g \in G(\mathbb{A}). \tag{2.4}$$

The Eisensetein series $E(s, \widetilde{\phi}, g, P)$ converges absolutely for sufficiently large $\mathrm{Re}(s) \gg 0$ and extends to a meromorphic function of s on $\mathbb{C}$, with a finite number of poles in the plane $\mathrm{Re}(s) > 0$, all simple and on the real axis (See [58]).

Let W be the Weyl group of A_0 in G. We denote the subset of Δ which generates M by θ. Then $\Delta = \theta \cup \{\alpha\}$. Then there exists a unique element $\widetilde{w} \in W$ such that $\widetilde{w}(\theta) \subset \Delta$ and $\widetilde{w}(\alpha) \in \Phi^-$. Fix a representative $w \in K_{\mathbb{A}} \cap G(F)$ for $\widetilde{w}$. We shall also denote every component of w by w again.

Let

$$I(s, \pi) = \mathrm{Ind}_{M(\mathbb{A})N(\mathbb{A})}^{G(\mathbb{A})} \pi \otimes \exp\langle s\tilde{\alpha}, H_P(\,\cdot\,)\rangle \otimes 1$$

be the representation of $G(\mathbb{A})$ induced from $P(\mathbb{A})$. Then $I(s, \pi) = \otimes_v I(s, \pi_v)$ with

$$I(s, \pi_v) = \mathrm{Ind}_{M_v N_v}^{G_v} \pi_v \otimes q_v^{\langle s\tilde{\alpha}, H_P(\,\cdot\,)\rangle} \otimes 1,$$

where q_v should be replaced by exp if v is archimedean. We let M' be the subgroup of G generated by $\widetilde{w}(\theta)$. Then there is a parabolic subgroup $P' \supset B$ with $P' = M'N'$.

Here M' is a Levi factor of P' and N' is the unipotent radical of P'. For $f \in I(s,\pi)$ and sufficiently large $\mathrm{Re}(s) \gg 0$, we define

$$M(s,\pi)f(g) = \int_{N'(\mathbb{A})} f(w^{-1}ng)dn, \quad g \in G(\mathbb{A}). \tag{2.5}$$

At each place v, for sufficiently large $\mathrm{Re}(s) \gg 0$, we define a local intertwining operator by

$$A(s,\pi_v,w)f_v(g) = \int_{N'_v} f_v(w^{-1}ng)dn, \quad g \in G_v, \tag{2.6}$$

where $f_v \in I(s,\pi_v)$. Then

$$M(s,\pi) = \otimes_v A(s,\pi_v,w). \tag{2.7}$$

It follows from the theory of Eisenstein series that for $\mathrm{Re}(s) \gg 0$, $M(s,\pi)$ extends to a meromorphic function of $s \in \mathbb{C}$ with only a finite number of simple poles (cf. [58]).

Let ${}^LM$ and ${}^LN$ be the Levi factor and the unipotent radical of the parabolic subgroup ${}^LP = {}^LM{}^LN$ of the L-group ${}^LG$. Then we have the representation $r : {}^LM \longrightarrow \mathrm{End}\big({}^L\mathfrak{n}\big)$ given by the adjoint action of ${}^LM$ on the Lie algebra ${}^L\mathfrak{n}$ of ${}^LN$. Let

$$r = r_1 \oplus r_2 \oplus \cdots \oplus r_m$$

be the decomposition of r into irreducible constituents. Each irreducible constituent (r_i, V_i) with $1 \le i \le m$ is characterized by

$$V_i = \left\{ X_{\beta^\vee} \in {}^L\mathfrak{n} \,|\, \langle \tilde{\alpha}, \beta \rangle = i \right\}, \quad i = 1, \cdots, m.$$

We refer to [55] and [77, Proposition 4.1] for more detail.

According to [53] and [55], one has

$$M(s,\pi)f = \Big(\otimes_{v\in S} A(s,\pi_v,w)f_v \Big) \bigotimes \Big(\otimes_{v\notin S} \tilde{f}_v \Big) \times \prod_{i=1}^{m} \frac{L_S(is,\pi,r_i)}{L_S(1+is,\pi,r_i)}, \tag{2.8}$$

where $f = \otimes_v f_v$ is an element in $I(s,\pi)$ such that for each $v \notin S$, f_v is the unique K_v-fixed vector with $f_v(e_v) = 1$, $\tilde{f}_v$ is the K_v-fixed vector in $I(-s, \widetilde{w}(\pi_v))$ with $\tilde{f}_v(e_v) = 1$, and $\widetilde{r}_i$ denotes the contragredient of r_i $(1 \le i \le m)$.

For every archimedean place v of F, let $\varphi_v : W_{F_v} \longrightarrow {}^LM_v$ be the corresponding homomorphism (cf. [63]) attached to π_v. One has a natural homomorphism $\eta_v : {}^LM_v \longrightarrow {}^LM$. We put

$$r_{i,v} = r_i \circ \eta_v, \quad i = 1, 2, \cdots, m.$$

Then $r_{i,v} \circ \varphi_v = r_i \circ \eta_v \circ \varphi_v$ is a finite dimensional representation of the Weil group W_{F_v} on V_i. Let $L(s, r_{i,v} \circ \varphi_v)$ be the corresponding Artin L-function attached to $r_{i,v} \circ \varphi_v$ (cf. [57]). We set

$$L^S(s, \pi, r_i) = \prod_{v=\infty} L(s, r_{i,v} \circ \varphi_v) \cdot \prod_{\substack{v \notin S \\ v < \infty}} L(s, \pi_v, r_i \circ \eta_v). \tag{2.9}$$

Let $\rho : M(\mathbb{A}) \longrightarrow \overline{M}(\mathbb{A})$ be the projection of $M(\mathbb{A})$ onto its adjoint group.

Shahidi showed the following.

Theorem 2.1(Shahidi [77]). *Let $\pi = \otimes_v \pi_v$ be a cuspidal automorphic representation of $\overline{M}(\mathbb{A})$. Then every L-function $L^S(s, \pi, r_i \circ {}^L\rho)$, $1 \le i \le m$, extends to a meromorphic function of s to the whole complex plane. Moreover, if π is generic, then each $L^S(s, \pi, r_i \circ {}^L\rho)$ satisfies a standard functional equation, that is,*

$$L^S(s, \pi, r_i \circ {}^L\rho) = \varepsilon_S(s, \pi, r_i \circ {}^L\rho)\, L^S(1 - s, \pi, \widetilde{r_i \circ {}^L\rho}),$$

where $\varepsilon_S(1s, \pi, r_i \circ {}^L\rho)$ is the root number attached to π and $r_i \circ {}^L\rho$, and $\widetilde{\tau}$ denotes the contragredient of a representation τ.

Furthermore, for a given generic cuspidal automorphic representation $\pi = \otimes_v \pi_v$ of $M(\mathbb{A})$, Shahidi defined the local L-functions $L(s, \pi_v, r_i)$ and the local root numbers $\varepsilon(s, \pi_v, r_i, \psi)$ with $1 \le i \le m$ at *bad places* v so that the completed L-function $L(s, \pi, r_i)$ and the completed root number $\varepsilon(s, \pi, r_i)$ defined by

$$L(s, \pi, r_i) = \prod_{\text{all } v} L(s, \pi_v, r_i), \quad \varepsilon(s, \pi, r_i) = \prod_{\text{all } v} \varepsilon(s, \pi_v, r_i, \psi), \quad i = 1, \cdots, m$$

satisfy a standard functional equation

$$L(s, \pi, r_i) = \varepsilon(s, \pi, r_i)\, L(1 - s, \pi, \widetilde{r_i}), \quad i = 1, \cdots, m. \tag{2.10}$$

Example 2.2(Kim-Shahidi [45]). Let F be a number field and let G be a simply connected semisimple split group of type G_2 over F. We set $\mathbb{A}_\infty = \prod_{v=\infty} F_v$. Let K_∞ be the standard maximal compact subgroup of $G(\mathbb{A}_\infty)$ and $K_v = G(\mathfrak{o}_v)$ for a finite place v. Then $K_{\mathbb{A}} = K_\infty \times \prod_{v<\infty} K_v$ is a maximal compact subgroup of $G(\mathbb{A})$. Fix a split maximal torus T in G and let $B = TU$ be a Borel subgroup of G. In what follows the roots are those of T in U. Let $\Delta = \{\beta_1, \beta_6\}$ be a basis of the root system Φ with respect to (T, B) with the long simple root β_1 and the short one β_6. Then the other roots are given by

$$\beta_2 = \beta_1 + \beta_6, \quad \beta_3 = 2\beta_1 + 3\beta_6, \quad \beta_4 = \beta_1 + 2\beta_6, \quad \beta_5 = \beta_1 + 3\beta_6.$$

Let P be the maximal parabolic subgroup of G generated by β_1 with Levi decomposition $P = MN$, where $M \simeq GL(2)$. See [80, Lemma 2.1]. Thus one has

$$\mathfrak{a}^* = \mathbb{R}\,\beta_4, \quad \mathfrak{a} = \mathbb{R}\,\beta_4^\vee \quad \text{and} \quad \rho_P = \frac{5}{2}\,\beta_4.$$

Let $\widetilde{\alpha} = \beta_4$. Then $s\widetilde{\alpha}\,(s \in \mathbb{C})$ corresponds to the character $|\det(m)|^s$. We note that $\mathbb{A}^* = \mathbb{A}_1^* \cdot \mathbb{R}_+^*$, where $\mathbb{A}_1^*$ is the group of ideles of norm 1. Let $\pi = \otimes_v \pi_v$ be a cuspidal automorphic representation of $M(\mathbb{A}) = GL(2, \mathbb{A})$. We may and will assume that the central character ω_π of π is trivial on $\mathbb{R}_+^*$. For a K-finite function φ in the representation space of π, the Eisenstein series $E(s, \widetilde{\varphi}, g) = E(s, \widetilde{\varphi}, g, P)$ defined by Formula (2.4) converges absolutely for sufficiently large $\mathrm{Re}(s) \gg 0$ and extends to a meromorphic function of s on $\mathbb{C}$, with a finite number of poles in the plane $\mathrm{Re}(s) > 0$, all simple and on the real axis. The discrete spectrum $L^2_{disc}\big(G(F)\backslash G(\mathbb{A})\big)_{(M,\pi)}$ is spanned by the residues of Eisenstein series for $\mathrm{Re}(s) > 0$ (See [58]). We know that the poles of Eisenstein series coincide with those of its constant terms. So it is enough to consider term along P. For each $f \in I(s, \pi)$, the constant term of $E(s, f, g)$ along P is given by

$$E_0(s, f, g) = \sum_{w \in \Omega} M(s, \pi, w) f(g), \quad \Omega = \{1, s_6 s_1 s_6 s_1 s_6\},$$

where s_i is the reflection along β_i defined by

$$s_i(\beta) = \beta - \frac{2(\beta_i, \beta)}{(\beta_i, \beta_i)}\beta_i, \quad 1 \le i \le 6,\ \beta \in \Phi.$$

Weyl group representatives are all chosen to lie in $K_{\mathbb{A}} \cap G(F)$. Here

$$M(s, \pi, w) f(g) = \int_{N_w^-(\mathbb{A})} f(w^{-1} n g)\, dn = \prod_v \int_{N_w^-(F_v)} f_v(w_v^{-1} n_v g_v)\, dn_v,$$

where $g = \otimes_v g_v \in G(\mathbb{A})$, $f = \otimes_v f_v$ is an element of $I(s, \pi)$ such that f_v is the unique K_v-fixed function normalized by $f_v(e_v) = 1$ for almost all v, and

$$N_w^- = \prod_{\substack{\alpha > 0 \\ w^{-1}\alpha < 0}} U_\alpha, \quad U_\alpha = \text{the one parameter unipotent subgroup.}$$

Let $\mathrm{St} : GL(2, \mathbb{C}) \longrightarrow GL(2, \mathbb{C})$ be the standard representation of $GL(2, \mathbb{C})$ and

$$\mathrm{Sym}^3 : GL(2, \mathbb{C}) = {}^L M \longrightarrow GL(4, \mathbb{C})$$

be the third symmetric power representation of $GL(2, \mathbb{C})$. Let

$$\big(\mathrm{Sym}^3\big)^0 = \mathrm{Sym}^3 \bigotimes \big(\wedge^2 \mathrm{St}\big)^{-1}$$

be the adjoint cube representation of $GL(2, \mathbb{C})$ (cf. [80, p. 249]). Then the adjoint representation r of ${}^L M = GL(2, \mathbb{C})$ on the Lie algebra ${}^L\mathfrak{n}$ of ${}^L N$ is given by

$$r = \big(\mathrm{Sym}^3\big)^0 \oplus \wedge^2 \mathrm{St}.$$

According to Formula (2.8), one obtain, for $w = s_6 s_1 s_6 s_1 s_6$, the formula

$$\begin{aligned} M(s,\pi)f = \quad & \Big(\otimes_{v\in S} M(s,\pi_v,w) f_v \Big) \bigotimes \Big(\otimes_{v\notin S} \tilde{f}_v \Big) \\ & \times L_S\big(s,\widetilde{\pi},(\mathrm{Sym}^3)^0\big)\, L_S\big(2s,\widetilde{\pi},\wedge^2\mathrm{St}\big) \\ & \times L_S\big(1+s,\widetilde{\pi},(\mathrm{Sym}^3)^0\big)^{-1}\, L_S\big(1+2s,\widetilde{\pi},\wedge^2\mathrm{St}\big)^{-1}, \end{aligned}$$

where S is a finite set of places of F including all the archimedean places such that π_v is unramifies for every $v \notin S$. Here $L_S\big(s,\pi,\wedge^2\mathrm{St}\big)$ is the partial Hecke L-function. Kim and Shahidi [45] proved that if π is a non-monomial cuspidal representation of $M(\mathbb{A}) = GL(2,\mathbb{A})$ in the sense that $\pi \ncong \pi\otimes\eta$ for any nontrivial grossencharacter η of $F^*\backslash\mathbb{A}_F^+$, the symmetric cube L-function $L\big(s,\pi,\mathrm{Sym}^3\big)$ and the adjoint cube L-function $L\big(s,\pi,(\mathrm{Sym}^3)^0\big)$ are both *entire* and satisfy the standard functional equations

$$L\big(s,\pi,\mathrm{Sym}^3\big) = \varepsilon\big(s,\pi,\mathrm{Sym}^3\big)\, L\big(1-s,\widetilde{\pi},\mathrm{Sym}^3\big)$$

and

$$L\big(s,\pi,(\mathrm{Sym}^3)^0\big) = \varepsilon\big(s,\pi,(\mathrm{Sym}^3)^0\big)\, L\big(1-s,\widetilde{\pi},(\mathrm{Sym}^3)^0\big).$$

It follows from this fact that if π is not monomial, the partial Rankin triple L-function $L_S(s,\pi\times\pi\times\pi)$ is entire. Ikeda [32] calculated the poles of the Rankin triple L-function $L_S(s,\pi\times\pi\times\pi)$ for $GL(2)$. And we have the following relations

$$L_S(s,\pi\times\pi\times\pi) = L\big(s,\pi,\mathrm{Sym}^3\big)\,\big(L_S(s,\pi\otimes\omega_\pi)\big)^2 \tag{2.11}$$

and

$$L\big(s,\pi,\mathrm{Sym}^3\big) = L_S\big(s,\pi\otimes\omega_\pi,(\mathrm{Sym}^3)^0\big). \tag{2.12}$$

According to Formula (2.10), $L_S(s,\pi\times\pi\times\pi)$ could have double zeros at $s = 1/2$.

In [47], Kim and Shahidi studied the cuspidality of the symmetric fourth power $\mathrm{Sym}^4(\pi)$ of a cuspidal representation π of $GL(2,\mathbb{A})$ and the partial symmetric m-th power L-functions $L_S\big(s,\pi,\mathrm{Sym}^m\big)$ $(1\le m\le 9)$. For the definition of $\mathrm{Sym}^m(\pi)$, we refer to Example 5.6 in this article. We summarize their results.

Theorem 2.3(Kim-Shahidi [47]). *Let π be a cuspidal automorphic representation of $GL(2,\mathbb{A})$ with ω_π its central character. Then $\mathrm{Sym}^4(\pi)\otimes\omega_\pi^{-1}$ is a cuspidal representation of $GL(5,\mathbb{A})$ except for the following three cases:*
(1) *π is monomial in the sense that $\pi\cong\pi\otimes\eta$ for some nontrivial Grössencharacter η of F.*
(2) *π is not monomial and $\mathrm{Sym}^3(\pi)\otimes\omega_\pi^{-1}$ is not cuspidal.*
(3) *$\mathrm{Sym}^3(\pi)\otimes\omega_\pi^{-1}$ is cuspidal and there exists a nontrivial quadratic character χ such that*
$\mathrm{Sym}^3(\pi)\otimes\omega_\pi^{-1}\cong\mathrm{Sym}^3(\pi)\otimes\omega_\pi^{-1}\otimes\chi$.

As applications of Theorem 2.3, they obtained the following.

Proposition 2.4(Kim-Shahidi [47]). *Let π be a cuspidal automorphic representation of $GL(2,\mathbb{A})$ with ω_π its central character such that $\mathrm{Sym}^3(\pi)$ is cuspidal. Then the following statements hold:*
(a) *Each partial symmetric m-th power L-functions $L_S\big(s,\pi,\mathrm{Sym}^m\big)$ $(m=6,7,8,9)$ has a meromorphic continuation and satisfies a standard functional equation.*
(b) $L_S\big(s,\pi,\mathrm{Sym}^5\big)$ *and* $L_S\big(s,\pi,\mathrm{Sym}^7\big)$ *are holomorphic and nonzero for* $\mathrm{Re}(s)\geq 1$.
(c) *If* $\omega_\pi^3=1$, $L_S\big(s,\pi,\mathrm{Sym}^6\big)$ *is holomorphic and nonzero for* $\mathrm{Re}(s)\geq 1$.
(d) *If* $\mathrm{Sym}^4(\pi)$ *is cuspidal,* $L_S\big(s,\pi,\mathrm{Sym}^6\big)$ *is holomorphic and nonzero for* $\mathrm{Re}(s)\geq 1$.
(e) *If* $\mathrm{Sym}^4(\pi)$ *is cuspidal and* $\omega_\pi^4=1$, $L_S\big(s,\pi,\mathrm{Sym}^8\big)$ *is holomorphic and nonzero for* $\mathrm{Re}(s)\geq 1$.
(f) *If* $\mathrm{Sym}^4(\pi)$ *is cuspidal,* $L_S\big(s,\pi,\mathrm{Sym}^9\big)$ *has a most a simple pole or a simple zero at* $s=1$.
(g) *If* $\mathrm{Sym}^4(\pi)$ *is not cuspidal,* $L_S\big(s,\pi,\mathrm{Sym}^9\big)$ *is holomorphic and nonzero for* $\mathrm{Re}(s)\geq 1$.

Proposition 2.5(Kim-Shahidi [47]). *Let $\pi=\otimes_v\pi_v$ be a cuspidal automorphic representation of $GL(2,\mathbb{A})$ such that $\mathrm{Sym}^3(\pi)$ is cuspidal. Let $\mathrm{diag}(\alpha_v,\beta_v)$ be the Satake parameter for π_v. Then $|\alpha_v|,|\beta_v|<q_v^{1/9}$. If $\mathrm{Sym}^4(\pi)$ is not cuspidal, then $|\alpha_v|=|\beta_v|=1$, that is, the full Ramanujan conjecture holds.*

Proposition 2.6(Kim-Shahidi [47]). *Let $\pi=\otimes_v\pi_v$ be a nonmonomial cuspidal automorphic representation of $GL(2,\mathbb{A})$ with a trivial central character. Suppose $m\leq 9$. Then the following statements hold:*
(1) *Suppose $\mathrm{Sym}^3(\pi)$ is not cuspidal. Then $L_S\big(s,\pi,\mathrm{Sym}^m\big)$ is holomorphic and nonzero at $s=1$, except for $m=6,8$; the L-functions $L_S\big(s,\pi,\mathrm{Sym}^6\big)$ and $L_S\big(s,\pi,\mathrm{Sym}^8\big)$ each have a simple pole at $s=1$.*
(2) *Suppose $\mathrm{Sym}^3(\pi)$ is cuspidal but $\mathrm{Sym}^4(\pi)$ is not cuspidal. Then $L_S\big(s,\pi,\mathrm{Sym}^m\big)$ is holo-morphic and nonzero at $s=1$ for $m=1,\cdots,7$ and $m=9$; the L-function $L_S\big(s,\pi,\mathrm{Sym}^8\big)$ has a simple pole at $s=1$.*

We are still far from solving Conjecture A. We have two known methods to study analytic properties of automorphic L-functions. The first is the method of constructing explicit zeta integrals that is called the Rankin-Selberg method. The second is the so-called Langlands-Shahidi method I just described briefly. In the late 1960s Langlands [55] recognized that many automorphic L-functions occur in the constant terms of the Eisenstein series associated to cuspidal automorphic representations of the Levi subgroups of maximal parabolic subgroups of split reductive groups through his intensively deep work on the theory of Eisenstein series. He obtained some analytic properties of certain automorphic L-functions using the meromorphic continuation and the functional equation of Eisenstein series. As mentioned above, he proved the meromorphic continuation of certain class of L-functions but

did not gave an answer to the functional equation. Shahidi [77] generalized Langlands' recognition to quasi-split groups, and calculated non-constant terms of the Eisenstein series and hence obtained the functional equations of a more broader class of many automorphic L-functions. In fact, those L-functions dealt with by Shahidi include most of the L-functions studied by other mathematicians (cf. [23], [24]). The first method has some advantage to provide more precise information on the location of poles and the special values of automorphic L-functions. On the other hand, the Langlands-Shahidi method has been applied to a large class of automorphic L-functions, and is likely to be more suited to the theory of harmonic analysis on a reductive group. Moreover the second method plays an important role in investigating the non-vanishing of automorphic L-functions on the line $\mathrm{Re}(s) = 1$. One of the main contributions of Shahidi to the Langlands-Shahidi method is to define local L-functions even at bad places in such a way that the completed L-function satisfies the functional equation. We are in need of new methods to have more knowledge on the analytic and arithmetic properties of automorphic L-functions.

3. Local Langlands conjecture

Let k be a local field and let W_k be its Weil group. We review the definition of the Weil group W_k following the article of Tate (cf. [82]). If $k = \mathbb{C}$, then $W_{\mathbb{C}} = \mathbb{C}^\times$. If $k = \mathbb{R}$, then

$$W_{\mathbb{R}} = \mathbb{C}^* \cup \tau\mathbb{C}^*, \quad \tau z \tau^{-1} = \overline{z},$$

where $z \mapsto \overline{z}$ is the nontrivial element of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Then $W_{\mathbb{R}}^{\mathrm{ab}} = \mathbb{R}^*$. Here if Y^c denotes the closure of the commutator subgroup of a topological group Y, we set $Y^{\mathrm{ab}} = Y/Y^c$.

Suppose k is a nonarchimedean local field and $\overline{k}$ a separable algebraic closure of k. Let q be the order of the residue field κ of k. We set $\Gamma_k = \mathrm{Gal}\big(\overline{k}/k\big)$ and $\Gamma_\kappa = \mathrm{Gal}\big(\overline{\kappa}/\kappa\big)$. Let $\Phi_\kappa : x \mapsto x^q$ be the Frobenius automorphism in Γ_κ. We set $\langle \Phi_\kappa \rangle = \{ \Phi_\kappa^n \mid n \in \mathbb{Z} \}$. Let $\varphi : \Gamma_k \longrightarrow \Gamma_\kappa$ be the canonical surjective homomorphism given by $\sigma \mapsto \sigma|_{\overline{\kappa}}$. The Weil group W_k is defined to be the set $W_k = \varphi^{-1}\big(\langle \Phi_\kappa \rangle\big)$. Obviously one has an exact sequence

$$1 \longrightarrow I_k \longrightarrow W_k \longrightarrow \langle \Phi_\kappa \rangle \longrightarrow 1,$$

where $I_k = \ker \varphi$ is the inertia group of k. We recall that W_k is topologized such that I_k has the induced topology from Γ_k, I_k is open in Γ_k and multiplication by Φ is a homeomorphism. Here Φ denotes a choice of a geometric Frobenius element in $\varphi^{-1}(\Phi_\kappa) \subset \Gamma_k$. We note that we have a continuous homomorphism $W_k \longrightarrow \Gamma_k$ with dense image. According to the local class field theory, one has the isomorphism

$$k^* \cong W_k^{\mathrm{ab}}. \tag{3.1}$$

In order to generalize the isomorphism (3.1) for $GL(1)$ to $GL(2)$, P. Deligne [18] introduced the so-called *Weil-Deligne group* W_k'. It is defined to be the group

scheme over $\mathbb{Q}$ which is the semidirect product of W_k by the additive group $\mathbb{G}_a$ on which W_k acts by the rule $wxw^{-1} = ||w||x$. We refer to [82, p. 19] for the definition of $||w||$. Namely, $W_k' = W_k \ltimes \mathbb{G}_a$ is the group scheme over $\mathbb{Q}$ with the multiplication

$$(w_1, a_1)(w_2, a_2) = (w_1 w_2, a_1 + ||w_1||\, a_2), \quad w_1, w_2 \in W_k,\ a_1, a_2 \in \mathbb{G}_a.$$

Definition 3.1(Deligne [19])**.** Let E be field of characteristic 0. A representation of W_k' over E is a pair $\rho' = (\rho, N)$ consisting of
(a) A finite dimensional vector space V over E and a homomorphism $\rho : W_k \longrightarrow GL(V)$ whose kernel contains an open subgroup of I_k, i.e., which is continuous for the discrete topology on $GL(V)$.
(b) A nilpotent endomorphism N of V such that

$$\rho(w) N \rho(w)^{-1} = ||w||\, N, \quad w \in W_k.$$

We see that a homomorphism of group schemes over E

$$\rho' : W_k \times_{\mathbb{Q}} E \longrightarrow GL(V)$$

determines, and is determined by a pair (ρ, N) as in Definition 3.1 such that

$$\rho'((w, a)) = \exp(aN)\, \rho(w), \quad w \in W_k,\ a \in \mathbb{G}_a.$$

Let $\rho' = (\rho, N)$ be a representation of W_k' over E. Define $\nu : W_k \longrightarrow \mathbb{Z}$ by $||w|| = q^{-\nu(w)},\ w \in W_k$. Then according to [19, (8.5)], there is a *unique unipotent* automorphism u of V such that

$$uN = Nu, \quad u\rho(w) = \rho(w)u, \quad w \in W_k$$

and such that

$$\exp(aN)\, \rho(w)\, u^{-\nu(w)}$$

is a semisimple automorphism of V for all $a \in E$ and $w \in W_k - I_k$. Then $\rho' = (\rho u^{-\nu}, N)$ is called the Φ-semisimplication of ρ'. And ρ' is called Φ-semisimple if and only if $\rho' = \rho'_{ss},\ u = 1$, i.e., the Frobeniuses acts semisimply.

Let $\rho' = (\rho, N, V)$ be a representation of W_k' over E. We let $V_N^I := (\ker N)^{I_k}$ be the subspace of I_k-invariants in $\ker N$. We define a local L-factor by

$$(3.2)\quad Z(t, V) = \det\left(1 - t\, \rho(\Phi)\big|_{V_N^I}\right)^{-1}, \ \text{ and } \quad L(s, V) = Z(q^{-s}, V), \ \text{when } E \subset \mathbb{C}.$$

We note that if $\rho' = (\rho, N)$ is a representation of W_k', then ρ' is irreducible if and only if $N = 0$ and ρ is irreducible.

Let G be a connected reductive group over a local field. A homomorphism $\alpha : W_k' \longrightarrow {}^L G$ is said to be *admissible* [9, p. 40] if the following conditions (i)-(iii)

are satisfied
(i) α is a homomorphism over Γ_k, i.e., the following diagram is commutative:

$$\begin{array}{ccc} W'_k & \xrightarrow{\alpha} & {}^L G \\ & \searrow \quad \swarrow & \\ & \Gamma_k & \end{array}$$

(ii) α is continuous, $\alpha(\mathbb{G}_a)$ are unipotent in ${}^L G^0$, and α maps semisimple elements into semisimple elements in ${}^L G$. Here an element x is said to be *semisimple* if $x \in I_k$, and an element $g \in {}^L G$ is called *semisimple* if $r(g)$ is semisimple for any finite dimensional representation r of ${}^L G$.
(iii) If $\alpha(W'_k)$ is contained in a Levi subgroup of a proper parabolic subgroup P of ${}^L G$, then P is relevant. See [9, p. 32].

Let $\mathcal{G}_k(G)$ be the set of all admissible homomorphisms $\phi : W'_k \longrightarrow {}^L G$ modulo inner automorphisms by elements of ${}^L G^0$. We observe that we can associate canonically to $\phi \in \mathcal{G}_k(G)$ a character χ_ϕ of the center $C(G)$ of G (cf. [9, p. 43], [63]). Let $Z_L^0 = C({}^L G^0)$ be the center of ${}^L G^0$. Following [9, pp. 43-44] and [63], we can construct a character ω_α of $G(k)$ associated to a cohomology class $\alpha \in H^1(W'_k, Z_L^0)$. If we write $\phi \in \mathcal{G}_k(G)$ as $\phi = (\phi_1, \phi_2)$ with $\phi : W'_k \longrightarrow {}^L G^0$ and $\phi : W'_k \longrightarrow \Gamma_k$, then ϕ_1 defines a cocycle of W'_k in ${}^L G^0$, and the map $\phi \mapsto \phi_1$ yields an embedding $\mathcal{G}_k(G) \hookrightarrow H^1(W'_k, {}^L G^0)$. Then $H^1(W'_k, Z_L^0)$ acts on $H^1(W'_k, {}^L G^0)$ and this action leaves $\mathcal{G}_k(G)$ stable [9, p. 40].

Let $\prod(G(k))$ be the set of all equivalence classes of irreducible admissible representations of $G(k)$. The following conjecture gives an arithmetic parametrization of irreducible admissible representations of $G(k)$.

Local Langlands Conjecture[LLC]. Let k be a local field. Let $\mathcal{G}_k(G)$ and $\prod(G(k))$ be as above. Then there is a surjective map $\prod(G(k)) \longrightarrow \mathcal{G}_k(G)$ with finite fibres which partitions $\prod(G(k))$ into disjoint finite sets $\prod_\phi(G(k))$, simply $\prod_\phi$ called *L-packets* satisfying the following (i)-(v):
(i) If $\pi \in \prod_\phi$, then the central character χ_π of π is equal to χ_ϕ.
(ii) If $\alpha \in H^1(W'_k, Z_L^0)$ and ω_α is its associated character of $G(k)$, then

$$\prod\nolimits_{\alpha \cdot \phi} = \left\{ \pi\omega_\alpha \mid \pi \in \prod\nolimits_\phi \right\}.$$

(iii) One element of $\prod_\phi$ is square integrable modulo the center $C(G)$ of G if and only if all elements are square integrable modulo the center $C(G)$ of G if and only if $\phi(W'_k)$ is not contained in any proper Levi subgroup of ${}^L G$.
(iv) One element of $\prod_\phi$ is tempered if and only if all elements of $\prod_\phi$ are tempered if and only if $\phi(W_k)$ is bounded.
(v) If H is a connected reductive group over k and $\eta : H(k) \longrightarrow G(k)$ is a k-morphism with commutative kernel and cokernel, then there is a required compatibilty between decompositions for $G(k)$ and $H(k)$. More precisely, η induces a canonical map ${}^L\eta : {}^L G \longrightarrow {}^L H$, and if we set $\phi' = {}^L\eta \circ \phi$ for $\phi \in \mathcal{G}_k(G)$, then any $\pi \in \prod_\phi(G(k))$, viewed as an $H(k)$-module, decomposes into a direct sum of finitely

many irreducible admissible representations belonging to $\prod_{\phi'}(H(k))$.

Remark 3.2. (a) If k is archimedean, i.e., $k = \mathbb{R}$ or $\mathbb{C}$, [LLC] was solved by Langlands [63]. We also refer the reader to [1], [2], [49].
(b) In case k is non-archimedean, Kazhdan and Lusztig [40] had shown how to parametrize those irreducible admissible representations of $G(k)$ having an Iwahori fixed vector in terms of admissible homomorphisms of W'_k.
(c) For a local field k of positive characteristic $p > 0$, [LLC] was established by Laumon, Rapoport and Stuhler [67].
(d) In case $G = GL(n)$ for a non-archimedean local field k, [LLC] was established by Harris and Taylor [28], and by Henniart [31]. In both cases, the correspondence was established at the level of a correspondence between irreducible Galois representations and supercuspidal representations.
(e) Let k be a a non-archimedean local field of characteristic 0 and let $G = SO(2n+1)$ the split special orthogonal group over k. In this case, Jiang and Soudry [38], [39] gave a parametrization of *generic supercuspidal* representations of $SO(2n+1)$ in terms of admissible homomorphisms of W'_k. More precisely, there is a unique bijection of the set of conjugacy classes of all admissible, completely reducible, multiplicity-free, symplectic complex representations $\phi : W'_k \longrightarrow {}^L SO(2n+1) = Sp(2n, \mathbb{C})$ onto the set of all equivalence classes of irreducible generic supercuspidal representations of $SO(2n+1, k)$.

For $\pi \in \prod_\phi(G)$ with $\phi \in \mathcal{G}_k(G)$, if r is a finite dimensional complex representation of ${}^L G$, we define the L- and ε-factors

$$L(s, \pi, r) = L(s, r \circ \phi) \quad \text{and} \quad \varepsilon(s, \pi, r, \psi) = \varepsilon(s, r \circ \phi, \psi), \tag{3.3}$$

where $L(s, r \circ \phi)$ is the Artin-Weil L-function.

Remark 3.3. For a non-archimedean local field k, Deligne [18] gave the complete formulation of [LLC] for $GL(2)$. In [18], he utilized for the first time the Weil-Deligne group W'_k, which was introduced by him in [19], in the context of ℓ-adic representations, in order to obtain a correct formulation in the case of $GL(2)$ over a non-archimedean local field.

Remark 3.4. The representations in the L-packet $\prod_\phi$ are parametrized by the component group

$$C_\phi := S_\phi / Z_L S_\phi^0,$$

where S_ϕ is the centralizer of the image of ϕ in ${}^L G$, S_ϕ^0 is the identity component of S_ϕ, and Z_L is the center of ${}^L G$. We refer the reader to [5], [51] for more information on the L-packets.

Example 3.5. Let π be a spherical or unramified representation of $G(k)$ with a non-archimedean local field k. It is known that $\pi \hookrightarrow I(\chi)$ for a unique unramified quasi-character χ of a maximal torus $T(k)$ of $G(k)$. Then π determines a semi-simple

conjugate class $c(\pi) = \{t_\pi\} \subset {}^L T \subset {}^L G$. Then the Langlands' parameter ϕ_π for π is

$$\phi_\pi : k^* \longrightarrow {}^L T \subset {}^L G, \qquad \phi_\pi(\widetilde{\omega}) = t_\pi$$

such that

$$\phi_\pi(\widetilde{\omega}) = t_\pi \quad \text{and } \phi_\pi \text{ is trivial on } \mathcal{O}^*,$$

where $\widetilde{\omega}$ denotes a uniformizer in k.
If $\pi = \pi(\mu_1, \cdots, \mu_n)$ is a spherical representation of $GL_n(k)$ with unramified characters μ_i $(i = 1, \cdots, n)$ of k^*, the semi-simple conjugacy class $c(\pi)$ is given by

$$c(\pi) = \{ \operatorname{diag}(\mu_1(\widetilde{\omega}), \cdots, \mu_n(\widetilde{\omega}))\}$$

and the Langlands' parameter ϕ_π for π is

$$\phi_\pi : k^* \longrightarrow {}^L T \subset {}^L G = GL(n, \mathbb{C}), \qquad \phi_\pi(\widetilde{\omega}) = \operatorname{diag}(\mu_1(\widetilde{\omega}), \cdots, \mu_n(\widetilde{\omega})).$$

4. Global Langlands conjecture

Let k be a global field and $\mathbb{A}$ its ring of adeles. This section is based on Arthur's article [3].

As in the local case of Section 3, the global Langlands conjecture should be a nonabelian generalization of abelian global class field theory. When Deligne [19] recognized the need to introduce the Weil-Deligne group W'_k for the local Langlands correspondence for $GL(2)$, it was realized that there seemed to be no natural global version of W'_k. In fact, Γ_k, W_k and W'_k are too small to parameterize all automorphic representations of a reductive group. Thus in the 1970s Langlands [60] attempted to discover a *hypothetical* group L_k to replace the Weil-Deligne group W'_k. Nowadays it is believed by experts that this group L_k should be related to the equally *hypothetical motivic Galois* group $\mathcal{M}_k$ of k. The group L_k is often called the *hypothetical* (or *conjectual*) Langlands group or the *automorphic* Langlands group. The notion of L_k and $\mathcal{M}_k$ is still not clear.

The global Langlands conjecture can be written as follows.

Global Langlands Conjecture[GLC]. Automorphic representations of $G(\mathbb{A})$ can be parametrized by admissible homomorphisms $\phi : L_k \longrightarrow {}^L G$ required to have the following properties (1)-(4):
(1) There is an L-packet $\prod_\phi$ which consists of automorphic representations of $G(\mathbb{A})$ attached to ϕ.
(2) Each L-packet $\prod_\phi$ is a finite set.
(3) Any automorphic representation of $G(\mathbb{A})$ belongs to $\prod_\phi$ for a unique homomorphism ϕ.
(4) The $\prod_\phi$'s are disjoint.

We first consider the case that k is a function field. Drinfeld [21] formulated a version of the global Langlands conjecture for function fields relating the irreducible

two dimensional representations of the Galois group Γ_k with irreducible cuspidal representations of $GL(2,\mathbb{A})$, and established the global Langlands conjecture. In the early 2000s Lafforgue [52] had extended the work of Drinfeld mentioned above to $GL(n)$ to give a proof of the global Langlands conjecture for $GL(n)$ over a function field. The formulation of the global Langlands conjecture made by Drinfeld and Lafforgue is essentially the same as that in the local non-archimedean case discussed in Section 3 with a few modification. So we omit the details for the global Langlands conjecture over a function field here. We refer to [20], [21], [52] for more detail.

Next we consider the case k is a number field. We first recall that according to the global class field theory, for $n = 1$, there is a canonical bijection between the continuous characters of Γ_k and characters of finite order of the idele group $k^*\backslash\mathbb{A}^*$. We should replace Γ_k by the Weil group W_k in order to obtain all the characters of $k^*\backslash\mathbb{A}^*$. For $n \geq 2$, by analogy with the local Langlands conjecture, we need a global analogue of the Weil-Deligne group W'_k. However no such analogue is available at this moment. We hope that the hypothetical Langlands group L_k plays a role as W'_k and fits into an exact sequence

$$1 \longrightarrow L_k^c \longrightarrow L_k \longrightarrow \Gamma_k \longrightarrow 1, \tag{4.1}$$

where L_k^c is a complex pro-reductive group. L_k should be a locally compact group equipped with an embedding $i_v : L_{k_v} \longrightarrow L_k$ for each completion k_v of k. Let G be a connected, quasi-split reductive group over k. We set $G_v := G(k_v)$ for every place v of k. Let $\mathfrak{L}_k(G)$ be the set of all equivalence classes of continuous, completely reducible homomorphisms ϕ of L_k into ${}^LG$, and $\mathcal{A}_k(G)$ the set of equivalence classes of all automorphic representations of $G(\mathbb{A})$. For each place v of k, let $\mathfrak{L}_{k_v}(G)$ be the set of equivalence classes of continuous, completely reducible homomorphisms $\phi_v : L_{k_v} \longrightarrow {}^LG_v$ and $\mathcal{A}_{k_v}(G)$ the set of continuous irreducible admissible representations of G_v. We would hope to have a bijection

$$\mathfrak{L}_k(G) \longrightarrow \mathcal{A}_k(G), \qquad \phi \mapsto \pi_\phi. \tag{4.2}$$

Moreover the set $\mathfrak{L}_k^0(G)$ of equivalence classes of irreducible representations in $\mathfrak{L}_k(G)$ should be in bijective correspondence with the set $\mathcal{A}_k^0(G)$ of all *cuspidal* automorphic representations in $\mathcal{A}_k(G)$. This would be supplemented by local bijection

$$\mathfrak{L}_{k_v}(G) \longrightarrow \mathcal{A}_{k_v}(G) \quad \text{for any place } v \textit{ of } k. \tag{4.3}$$

The local and global bijections should be *compatible* in the sense that for any $\phi : L_k \longrightarrow {}^LG$, there is an automorphic representation $\pi_\phi = \otimes_v \pi_{\phi,v}$ of $G(\mathbb{A})$ with the correspondence $\phi \mapsto \pi_\phi$ such that for each place v of k, the restriction $\phi_v = \phi \circ i_v$ of ϕ to L_{k_v} corresponds to the local component $\pi_{\phi,v}$ of π_ϕ. Of course, one expects that all of these correspondences (4.2) and (4.3) would satisfy properties similar to those in the local Langlands conjecture, e.g., the preservation of L- and ε-factors with twists, etc.

The local Langlands groups are elementary. They are given by

$$L_{k_v} = \begin{cases} W_{k_v} & \text{if } v \text{ is archimedean,} \\ W_{k_v} \times SU(2,\mathbb{R}) & \text{if } v \text{ is nonarchimedean,} \end{cases}$$

where W_{k_v} is again the Weil group of k_v. Thus the local Langlands group L_{k_v} is a split extension of W_{k_v} by compact simply connected Lie group. But the hypothetical Langlands group will be much larger. It would be an infinite fibre product of nonsplit extension

$$1 \longrightarrow K_c \longrightarrow L_c \longrightarrow W_k \longrightarrow 1 \tag{4.4}$$

of the Weil group W_k by a compact, semsimple, simply connected Lie group K_c. However one would have to establish something in order to show that L_k has all the desired properties.

Grothendieck's conjectural theory of motives introduces the so-called *motivic Galois* group $\mathcal{M}_k$, which is a reductive proalgebraic group over $\mathbb{C}$ and comes with a proalgebraic projection $\mathcal{M}_k \longrightarrow \Gamma_k$. A *motive* of rank n is to be defined as a proalgebraic representation

$$\mathbb{M} : \mathcal{M}_k \longrightarrow GL(n,\mathbb{C}).$$

We observe that any continuous representation of Γ_k pulls back to $\mathcal{M}_k$ and can be viewed as a motive in the above sense. It is conjectured that the arithmetic information in any motive $\mathbb{M}$ is directly related to analytic information from some automorphic representations of $G(\mathbb{A})$. The conjectural theory of motives also applies to any completion k_v of k. It produces a proalgebraic group $\mathcal{M}_{k_v}$ over $\mathbb{C}$ that fits into a commutative diagram

$$\begin{array}{ccc} \mathcal{M}_{k_v} & \longrightarrow & \mathcal{M}_k \\ \downarrow & & \downarrow \\ \Gamma_{k_v} & \hookrightarrow & \Gamma_k \end{array}$$

of proalgebraic homomorphisms.

In 1979, Langlands [60] speculated the following:

Conjecture B(Langlands [60, Section 2])**.** There is a commutative diagram

$$\begin{array}{ccc} L_k & \xrightarrow{\phi} & \mathcal{M}_k \\ & \searrow \quad \swarrow & \\ & \Gamma_k & \end{array}$$

together with a compatible commutative diagram

$$\begin{array}{ccc} L_{k_v} & \xrightarrow{\phi_v} & \mathcal{M}_{k_v} \\ & \searrow \quad \swarrow & \\ & \Gamma_k & \end{array}$$

for each completion k_v of k, in which ϕ and ϕ_v are continuous homeomorphisms. There should be the analogue of the notion of *admissibility* of the maps ϕ and ϕ_v as in the local case (cf. [9, p. 40]).

The above conjecture implies that we can attach to any proalgebraic homomorphism μ from $\mathcal{M}_k$ to ${}^LG$ over Γ_k, its associated automorphic representation of $G(\mathbb{A})$. In particular, if we take $G = GL(n)$, it means that we can attach to any motive $\mathbb{M}$ of rank n, an automorphic representation $\pi_{\mathbb{M}} = \otimes_v \pi_{\mathbb{M},v}$ of $GL(n,\mathbb{A})$ with the following property: The family of semi-simple conjugacy classes $c(\pi_{\mathbb{M}}) = \{c(\pi_{\mathbb{M},v})\}$ in $GL(n,\mathbb{C})$ associated to $\pi_{\mathbb{M}}$ is equal to the family of conjugacy classes $c(\mathbb{M}) = \{c_v(\mathbb{M})\}$ obtained from $\mathbb{M}$, and the local homomorphism $\mathcal{M}_{k_v} \longrightarrow \mathcal{M}_k$ at places v that are unramified for $\mathbb{M}$. In fact, $c_v(\mathbb{M})$ is the image of the Frobenius class Fr_v under a different kind of Γ_k, namely a compatible family

$$\Gamma_k \longrightarrow \prod_{\ell \notin S(\mathbb{M}) \cup \{v\}} GL(n, \overline{\mathbb{Q}}_\ell)$$

of ℓ-adic representations attached to $\mathbb{M}$. Our task now is to find some natural ways to construct an explicit candidate for $\mathcal{M}_k$ and then to clarify the structure of $\mathcal{M}_k$. It is suggested by experts [70] that $\mathcal{M}_k$ be a proalgebraic fibre product of certain extensions

$$1 \longrightarrow \mathcal{D}_c \longrightarrow \mathcal{M}_c \longrightarrow \mathcal{T}_k \longrightarrow 1 \tag{4.5}$$

of a fixed group $\mathcal{T}_k$ by complex, semisimple, simply connected algebraic groups $\mathcal{D}_c$. The group $\mathcal{T}_k$ is an extension

$$1 \longrightarrow \mathcal{S}_k \longrightarrow \mathcal{T}_k \longrightarrow \Gamma_k \longrightarrow 1 \tag{4.6}$$

of Γ_k by a complex proalgebraic torus $\mathcal{S}_k$ (cf. [73, Chapter II], [60, Section 5], [74, Section 7]). The contribution to $\mathcal{M}_k$ of any $\mathcal{M}_c$ is required to match the contribution to L_k of a corresponding L_c, in which K_c is a compact real form of $\mathcal{D}_c$. This construction should have to come with the following diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & L_k^c & \longrightarrow & L_k & \longrightarrow & \Gamma_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathcal{M}_k^c & \longrightarrow & \mathcal{M}_k & \longrightarrow & \Gamma_k & \longrightarrow & 1 \end{array}$$

where $\mathcal{M}_k^c$ is a complex pro-reductive group.

Let $\Psi_k(G)$ be the set of equivalence classes of continuous, completely reducible homomorphisms of $\mathcal{M}_k$ into ${}^LG$, and for each place v of k, let $\Psi_{k_v}(G)$ be the set of equivalence classes of continuous, completely reducible homomorphisms of $\mathcal{M}_{k_v}$ into ${}^LG_v$. Then one should have to obtain a bijective correspondence

$$\Psi_k(G) \longrightarrow \mathcal{A}_k(G). \tag{4.7}$$

This would be supplemented by local bijective correspondences

$$\mathcal{M}_{k_v}(G) \longrightarrow \mathcal{A}_{k_v}(G) \tag{4.8}$$

for all places v of k.

5. Langlands functoriality

As we see in section 2, Shahidi [77] gave a partially affirmative answer to Conjecture A, which is a question raised by Langlands for a larger class of automorphic L-functions $L(s,\pi,r)$ obtained from cuspidal automorphic representations π of a Levi subgroup M of a quasi-split reductive group and the adjoint representation of ${}^LM$ on the real Lie algebra ${}^L\mathfrak{n}$ of ${}^LN$. This suggest trying, given a L-function and a quasi-split reductive group G, to see whether G has an automorphic representation with the given L-function. Many instances of such questions can be regarded as special cases of the *lifting problem*, nowadays called the *Principle of Functoriality*, with respect to morphisms of L-groups. The motivation of this problem stems from a global side. There is also a local version for this problem. These questions were raised and also formulated by Langlands [54] in the late 1960s. Roughly speaking, the principle of functoriality describes profound relationships among automorphic forms on different groups.

Let k be a local or global field, and let H, G two connected reductive groups defined over k. A homomorphism $\sigma : {}^LH \longrightarrow {}^LG$ is said to an *L-homomorphism* if it satisfies the following conditions (1)-(3):
(1) σ is a homomorphism over the absolute Galois group Γ_k, namely, the following diagram is commutative;

$$\begin{array}{ccc} {}^LH & \xrightarrow{\sigma} & {}^LG \\ & \searrow \quad \swarrow & \\ & \Gamma_k & \end{array}$$

(2) σ is continuous;
(3) The restriction of σ to ${}^LH^0$ is a complex analytic homomorphism of ${}^LH^0$ into ${}^LG^0$.

Let $\mathcal{G}_k(H)$ (resp. $\mathcal{G}_k(G)$) be the set of all admissible homomorphisms $\phi : W_k' \longrightarrow {}^LH$ (resp. ${}^LG$) modulo inner automorphisms by elements of ${}^LH^0$ (resp. ${}^LG^0$). Suppose G is quasi-split. Given a fixed L-homomorphism $\sigma : {}^LH \longrightarrow {}^LG$, if ϕ is any element in $\mathcal{G}_k(H)$, then the composition $\sigma \circ \phi$ is an element in $\mathcal{G}_k(G)$. It is easily seen that the correspondence $\phi \mapsto \sigma \circ \phi$ yields the canonical map

$$\mathcal{G}_k(\sigma) : \mathcal{G}_k(H) \longrightarrow \mathcal{G}_k(G).$$

If k is a global field and v is a place of k, then ${}^LG_v$ can be viewed as a subgroup of ${}^LG$ because Γ_{k_v} is regarded as a subgroup of Γ_k. Therefore σ induces the L-homomorphism $\sigma_v : {}^LH_v \longrightarrow {}^LG_v$ and hence a local map

$$\mathcal{G}_k(\sigma_v) : \mathcal{G}_k(H_v) \longrightarrow \mathcal{G}_k(G_v).$$

We refer to [9, pp. 54-58] for more detail on these stuffs.

Langlands Functoriality Conjecture(Langlands [54])**.** Let k be a global field,

and let H, G two connected reductive groups over k with G quasi-split. Suppose $\sigma : {}^LH \longrightarrow {}^LG$ is an L-homomorphism. Then for any automorphic representation $\pi = \otimes_v \pi_v$ of $H(\mathbb{A})$, there exists an automorphic representation $\Pi = \otimes_v \Pi_v$ of $G(\mathbb{A})$ such that

$$c(\Pi_v) = \sigma(c(\pi_v)), \quad v \notin S(\pi) \cup S(\Pi), \tag{5.1}$$

where $S(\pi)$ (resp. $S(\Pi)$) denotes a finite set of ramified places of k for π (resp. Π) so that π_v (resp. Π_v) is unramified for every place $v \notin S(\pi)$ (resp. $v \notin S(\Pi)$). We note that the condition (5.1) is equivalent to the condition

$$L_S(s, \Pi, r) = L_S(s, \pi, r \circ \sigma), \quad S = S(\pi) \cup S(\Pi) \tag{5.2}$$

for every finite dimensional complex representation r of ${}^LG$.

Remark 5.1. For a nonarchimedean local field k, we can formulate a local version of Langlands Functoriality Conjecture replacing the word "automorphic" by "admissible" and modifying some facts of an L-homomorphism.

Remark 5.2. Suppose k is a nonarchimedean local field with the ring of integers $\mathcal{O}$. Suppose H and G are quasi-split and there is a finite extension E of k such that both H and G split over E, and have an $\mathcal{O}$-structure so that both $H(\mathcal{O})$ and $G(\mathcal{O})$ are special maximal compact subgroups. Let π be an unramified representation of $H(k)$ in the sense that π has a nonzero $H(\mathcal{O})$-fixed vector, and let $\phi = \phi_\pi \in \mathcal{G}_k(H)$ be the unramified parameter of π. Then for any L-homomorphism $\sigma : {}^LH \longrightarrow {}^LG$, the parameter $\widetilde{\phi} = \sigma \circ \phi$ is unramified and defines an L-packet $\prod_{\widetilde{\phi}}(G)$ which contains exactly one unramified representation Π of $G(k)$ to be called the *natural lift* of π (cf. [9, p. 55]).

If we assume that the Local Langlands Conjecture, briefly [LLC] is valid, we can reformulate the Langlands Functoriality Conjecture using [LLC] in the following way. Let $\pi = \otimes_v \pi_v$ be an automorphic representation of $H(\mathbb{A})$. According to [LLC], we can attach to each π_v, an element $\phi_v : W'_k \longrightarrow {}^LH_v$ in $\mathcal{G}_{k_v}(H)$. The composition $\sigma \circ \phi_v$ is an element in $\mathcal{G}_{k_v}(G)$. By [LLC] again, one has an irreducible admissible representation Π_v of G_v attached to $\sigma \circ \phi_v$. Then $\Pi = \otimes_v \Pi_v$ is an irreducible admissible representation of $G(\mathbb{A})$. Therefore Langlands Functoriality Conjecture is equivalent to the statement that Π must be an automorphic representation of $G(\mathbb{A})$.

If we assume that Global Langlands Conjecture, briefly [GLC] is valid, we can also reformulate Langlands Functoriality Conjecture using [GLC] as follows: Given an automorphic representation π of $H(\mathbb{A})$ with its associated parameter $\phi_\pi : L_k \longrightarrow {}^LH$, there must be an L-packet $\Pi_{\sigma \circ \phi_\pi}(G)$ attached to $\sigma \circ \phi_\pi$.

Example 5.3. Suppose $H = \{1\}$ and $G = GL(n)$. Clearly an automorphic representation π of $H(\mathbb{A})$ is trivial. The choice of σ amounts to that of an admissible homomorphism

$$\sigma : \mathrm{Gal}(E/k) \longrightarrow GL(n, \mathbb{C}) = {}^LG$$

for a finite Galois extension E of k. Therefore Langlands Functoriality Conjecture reduces to the following assertion.

Strong Artin Conjecture(Langlands [54])**.** Let k be a number field. For an n-dimensional complex representation σ of $\mathrm{Gal}(E/k)$, there is an automorphic representation π of $GL(n, \mathbb{A})$ such that

$$c(\pi_v) = \sigma(Fr_v), \quad v \notin S(E),$$

where $S(E)$ is a finite set of places including all the ramified places of E. The above conjecture was established partially but remains unsettled for the most part. We summarize the cases that have been established until now chronically. (a) The case $n = 1$: This is the Artin reciprocity law, namely, $k^* \cong W_k^{\mathrm{ab}}$, which is the essential part of abelian class field theory. The image of σ is of cyclic type or of dihedral type.
(b) The case where $n = 2$ and $\mathrm{Gal}(E/k)$ is solvable: The conjecture was solved by Langlands [61] when the image of σ in $PSL(2, \mathbb{C})$ is of tetrahedral type, that is, isomorphic to A_4, and by Tunnell [84] when the image of σ in $PSL(2, \mathbb{C})$ is of octahedral type, i.e., isomorphic to S_4. It is a consequence of cyclic base change for $GL(2)$. These cases were used by A. Wiles [86] in his proof of Fermat's Last Theorem.
(c) The case where n is arbitrary and $\mathrm{Gal}(E/k)$ is nilpotent: The conjecture was established by Arthur and Clozel [6] as an application of cyclic base change for $GL(n)$.
(d) The case where $n = 2$ and the image of σ is of icosahedral type: Partial results were obtained by Taylor et al. (cf. [10]) C. Khare proved this case.
(e) The case where $n = 4$ and $\mathrm{Gal}(E/k)$ is solvable: The conjecture was established by Ramakrishnan [72] for representations σ that factor through the group $GO(4, \mathbb{C})$ of orthogonal similitudes.

Example 5.4. Let k be a number field. Let $H = Sp(2n)$, $SO(2n+1)$, $SO(2n)$ be the split form, and $G = GL(N)$, where $N = 2n+1$ or $2n$. Then ${}^L Sp(2n) = SO(2n+1, \mathbb{C})$, ${}^L SO(2n+1) = Sp(2n, \mathbb{C})$, ${}^L SO(2n) = SO(2n, \mathbb{C})$ and ${}^L GL(N) = GL(N, \mathbb{C})$. As an L-homomorphism $\sigma : {}^L H \longrightarrow {}^L G$, we take the embeddings

$${}^L Sp(2n) \hookrightarrow GL(2n+1, \mathbb{C}), \quad {}^L SO(2n+1) \hookrightarrow GL(2n, \mathbb{C}), \quad {}^L SO(2n) \hookrightarrow GL(2n, \mathbb{C}).$$

In each of these cases, the Langlands weak functorial lift for irreducible *generic* cuspidal automorphic representations of $H(\mathbb{A})$ was established by Cogdell, Kim, Piateski-Shapiro and Shahidi [13], [14]. Here the notion of "weak" automorphy means that an automorphic representation of $GL(n)$ exists whose automorphic L-function matches the desired Euler product except for a finite number of factors. The proof is based on the converse theorems for $GL(n)$ established by Cogdell and Piateski-Shapiro [15], [16]. It is still an open problem to establish the Langlands functorial lift from irreducible *non-generic* cuspidal automorphic representations of

$H(\mathbb{A})$ to $G(\mathbb{A})$. Let $H = \mathrm{GSpin}_m$ be the general spin group of semisimple rank $[\frac{m}{2}]$, i.e., a group whose derived group is Spin_m. Then the L-group of G is given by

$$ {}^L\mathrm{GSpin}_m = \begin{cases} GSO_m & \text{if } m \text{ is even;} \\ GSp_{2[\frac{m}{2}]} & \text{if } m \text{ is odd.} \end{cases} $$

In each case we have an embedding

$$ i : {}^L H^0 \hookrightarrow GL(N, \mathbb{C}), \quad N = m \text{ or } 2\Big[\frac{m}{2}\Big]. \tag{5.3} $$

Asgari and Shahidi [7], [8] proved that if π is a generic cuspidal representation of GSpin_m, then the functoriality is valid for the embedding (5.3).
If $H = SO(2n+1)$, for generic cuspidal representations, Jiang and Soudry [38] proved that the Langlands functorial lift from $SO(2n+1)$ to $GL(2n)$ is *injective* up to isomorphism. Using the functorial lifting from $SO(2n+1)$ to $GL(2n)$, Khare, Larsen and Savin [41] proved that for any prime ℓ and any even positive integer n, there are infinitely many exponents k for which the finite simple group $PSp_n(\mathbb{F}_{\ell^k})$ appears as a Galois group over $\mathbb{Q}$. Furthermore, in their recent paper [42] they extended their earlier work to prove that for a positive integer t, assuming that t is even if $\ell = 3$ in the first case (1) below, the following statements (1)-(3) hold:
(1) Let ℓ be a prime. Then there exists an integer k divisible by t such that the simple group $G_2(\mathbb{F}_{\ell^k})$ appears as a Galois group over $\mathbb{Q}$.
(2) Let ℓ be an odd prime. Then there exists an integer k divisible by t such that the simple finite group $SO_{2n+1}(\mathbb{F}_{\ell^k})^{\mathrm{der}}$ or the finite classical group $SO_{2n+1}(\mathbb{F}_{\ell^k})$ appears as a Galois group over $\mathbb{Q}$.
(3) If $\ell \equiv 3, 5 \pmod 8$ and ℓ is a prime, then there exists an integer k divisible by t such that the simple finite group $SO_{2n+1}(\mathbb{F}_{\ell^k})^{\mathrm{der}}$ appears as a Galois group over $\mathbb{Q}$. The construction of Galois groups in (1)-(3) is based on the functorial lift from $Sp(2n)$ to $GL(2n+1)$, and the backward lift from $GL(2n+1)$ to $Sp(2n)$ plus the theta lift from G_2 to $Sp(6)$.

Example 5.5. Let k be a number field. For two positive integers m and n, we let

$$ H = GL(m) \times GL(n) \qquad \text{and} \qquad G = GL(mn). $$

Then ${}^L H = GL(m, \mathbb{C}) \times GL(n, \mathbb{C})$ and ${}^L G = GL(mn, \mathbb{C})$. We take the L-homomorphism

$$ \sigma : GL(m) \times GL(n) \longrightarrow GL(mn, \mathbb{C}) $$

given by the tensor product. Suppose $\pi = \otimes_v \pi_v$ and $\tau = \otimes_v \tau_v$ are two cuspidal automorphic representations of $GL(m, \mathbb{A})$ and $GL(n, \mathbb{A})$ respectively. By [LLC] for $GL(N)$ [28], [31], [63], one has the parametrizations

$$ \phi_v : W_k' \longrightarrow GL(m, \mathbb{C}) \qquad \text{and} \qquad \psi_v : W_k' \longrightarrow GL(n, \mathbb{C}). $$

Let

$$ [\phi_v, \psi_v] : W_k' \longrightarrow GL(m, \mathbb{C}) \times GL(n, \mathbb{C}) \hookrightarrow GL(mn, \mathbb{C}) = {}^L G $$

be the admissible homomorphism of W_k' into ${}^L H$ defined by

$$[\phi_v, \psi_v](w) = \big(\phi_v(w), \psi_v(w)\big), \quad w \in W_k'.$$

The composition $\theta_v = \sigma \circ [\phi_v, \psi_v]$ is an admissible homomorphism of W_k' into ${}^L G$. According to [LLC] for $GL(N)$, one has an irreducible admissible representation of $GL(mn, k_v)$ attached to θ_v, denoted by $\pi_v \boxtimes \tau_v$. We set

$$\pi \boxtimes \tau = \bigotimes_v \pi_v \boxtimes \tau_v.$$

The validity of the Langlands Functoriality Conjecture for the L-homomorphism $\sigma : {}^L H \longrightarrow {}^L G$ implies that $\pi \boxtimes \tau$ is an automorphic representation of $GL(mn, \mathbb{A})$. Ramakrishnan [71] used the converse theorem for $GL(4)$ of Cogdell and Piatetski-Shapiro to establish the functoriality for $GL(2) \times GL(2)$. Kim and Shahidi [46] established the functoriality for $GL(2) \times GL(3)$.

Example 5.6. Let $H = GL(2)$. For a positive integer $m \geq 2$, let $G = GL(m+1)$. Suppose $\pi = \otimes_v \pi_v$ is an automorphic representation of $H(\mathbb{A})$. According to [LLC] for $GL(n)$ [28], [31], [63], for each place v of k, we have a semisimple conjugacy class $c(\pi_v) = \{\mathrm{diag}(\alpha_v, \beta_v)\} \subset GL(2, \mathbb{C})$. By [LLC] for $GL(n)$ again, for each place v of k, there is an irreducible admissible representation of $GL(m+1, k_v)$, denoted by $\mathrm{Sym}^m(\pi_v)$ attached to the semisimple conjugacy class

$$\big\{\mathrm{diag}\big(\alpha_v^m, \alpha_v^{m-1}\beta_v, \cdots, \beta_v^m\big)\big\} \subset GL(m+1, \mathbb{C}).$$

We set

$$\mathrm{Sym}^m(\pi) := \bigotimes_v \mathrm{Sym}^m(\pi_v).$$

Then $\mathrm{Sym}^m(\pi)$ is an irreducible admissible representation of $GL(m+1, \mathbb{A})$. The validity of the Langlands Functoriality Conjecture for the L-homomorphism $\mathrm{Sym}^m : GL(2, \mathbb{C}) \longrightarrow GL(m+1, \mathbb{C})$ implies that $\mathrm{Sym}^m(\pi)$ is an automorphic representation of $GL(m+1, \mathbb{A})$. As a consequence, we obtain a complete resolution of the Ramanujan-Petersson conjecture for Maass forms, the Selberg conjecture for eigenvalues and the Sato-Tate conjecture. In 1978, Gelbart and Jacquet [22] established the functoriality for Sym^2 using the converse theorem on $GL(3)$. In 2002, Kim and Shahidi [46] established the functoriality for Sym^3 using the functoriality for $GL(2) \times GL(3)$. Thereafter Kim [43] established the functoriality for Sym^4. The proof is based on the converse theorems for $GL(n)$ established by Cogdell and Piateski-Shapiro [15, 16]. We refer to [47] for more results on this topic.

Example 5.7. For a positive integer $n \geq 2$, we let

$$H = GL(n) \qquad \text{and} \qquad G = GL(N), \quad N = \frac{(n-1)n}{2}.$$

Let

$$\wedge^2 : {}^L H = GL(n, \mathbb{C}) \longrightarrow {}^L G = GL(N, \mathbb{C})$$

be the L-homomorphism of ${}^LH$ into ${}^LG$ given by the exterior square. Suppose $\pi = \otimes_v \pi_v$ is a cuspidal automorphic representation of $GL(n, \mathbb{A})$. According to [LLC] for $GL(m)$, for each place v of k, one has the admissible homomorphism

$$\phi_v : W_k' \longrightarrow {}^LH = GL(n, \mathbb{C})$$

parameterizing π_v. The composition $\psi_v = \wedge^2 \circ \phi_v$ again yields an irreducible admissible representation $\wedge^2 \pi_v$ of $GL(N, k_v)$ for every unramified representation π_v. We set

$$\wedge^2 \pi = \bigotimes_v \wedge^2 \pi_v.$$

Then $\wedge^2 \pi$ is an irreducible admissible representation of $GL(N, \mathbb{A})$. The validity of the Langlands functoriality implies that $\wedge^2 \pi$ is an automorphic representation of $GL(N, \mathbb{A})$. Kim [43] established the functoriality for the case $n = 4$, that is a functorial lift from $GL(4)$ to $GL(6)$.

Remark 5.8. As we see in Example 5.3, 5.4 and 5.5, the converse theorem for $GL(n)$ obtained by Cogdell and Piateski-Shapiro plays a crucial role in establishing the functoriality for $GL(2) \times GL(3)$, Sym^3 and Sym^4. There are widely known three methods in establishing the Langlands functoriality which are based on the theory of the Selberg-Arthur trace formula [4], [6], [61], [62], the converse theorems for $GL(n)$ [15], [16], [22] and the theta correspondence or theta lifting method (R. Howe, J. -S. Li, S. Kudla et al.).

According to the above examples and the converse theorems for $GL(n)$, we see that the importance of the Langlands Functoriality Conjecture is that automorphic L-functions of any automorphic representations of any group should be the L-functions of automorphic representations of $GL(n, \mathbb{A})$. In this sense we can say that $GL(n, \mathbb{A})$ is speculated to be the mother of all automorphic representations, and their offspring L-functions are already supposed to have meromorphic continuations and the standard functional equation.

Appendix : Converse theorems

We have seen that the converse theorems have been effectively applied to the establishment of the Langlands functoriality in certain special interesting cases (cf. Example 5.4, 5.5, 5.6 and 5.7). We understand that the converse theorems give a criterion for a given irreducible representation of $GL(n, \mathbb{A})$ to be *automorphic* in terms of the analytic properties of its associated automorphic L-functions. In this appendix, we give a brief survey of the history of the converse theorems and survey the recent results in the local converse theorems.

The first converse theorem was established by Hamburger [26] in 1921. This theorem states that any Dirichlet series satisfying the functional equation of the Riemann zeta function $\zeta(s)$ and suitable regularity conditions must be a multiple of $\zeta(s)$. More precisely, this theorem can be formulated:

Theorem A(Hamburger [26], 1921)**.** *Let two Dirichlet series* $g(s) = \sum_{n\geq 1} a_n n^{-s}$ *and* $h(s) = \sum_{n\geq 1} b_n n^{-s}$ *converge absolutely for* $\mathrm{Re}(s) > 1$. *Suppose that both* $(s-1)h(s)$ *and* $(s-1)g(s)$ *are entire functions of finite order. Assume we have the following functional equation:*

$$\pi^{-\frac{s}{2}}\,\Gamma(s/2)\,h(s) = \pi^{-\frac{1-s}{2}}\,\Gamma((1-s)/2)\,g(1-s).$$

Then $g(s) = h(s) = a_1\zeta(s)$. *Here* $\Gamma(s)$ *is the usual Gamma function, and an entire function* $f(s)$ *is said to be of order* ρ *if*

$$f(s) = O\big(|s|^{\rho+\epsilon}\big) \qquad for\ any \qquad \epsilon > 0.$$

Unfortunately Hamburger's converse theorem was not well recognized until the generalization to L-functions attached to holomorphic modular forms was done by Hecke [29] in 1936. Hecke proved his converse theorem connecting certain L-functions which satisfy a certain functional equation with holomorphic modular forms with respect to the full modular group $SL(2,\mathbb{Z})$. For a good understanding of Hecke's converse theorem, we need to describe Hecke's idea and argument roughly. Let

$$f(\tau) = \sum_{n\geq 1} a_n e^{2\pi i n\tau}$$

be a holomorphic modular form of weight d with respect to $SL(2,\mathbb{Z})$. To such a function f Hecke attached an L-function $L(s,f)$ via the Mellin transform

$$(2\pi)^{-s}\,\Gamma(s)\,L(s,f) = \int_0^\infty f(iy)\,y^s\,\frac{dy}{y}$$

and derived the functional equation for $L(s,f)$. He inverted this process by taking a Dirichlet series

$$D(s) = \sum_{n\geq 1} \frac{a_n}{n^s}$$

and assuming that it converges absolutely in some half plane, has an analytic continuation to an entire function of finite order, and satisfies the same functional equation as $L(s,f)$. In his masterpiece [29], he could reconstruct a holomorphic modular form from $D(s)$ by Mellin inversion

$$f(iy) = \sum_{n\geq 1} a_n e^{-2\pi n y} = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} (2\pi)^{-s}\,\Gamma(s)\,D(y)\,y^s\,ds$$

and obtain the modular transformation law for $f(\tau)$ under $\tau \mapsto -\tau^{-1}$ from the functional equation for $D(s)$ under $s \mapsto d-s$. This is Hecke's converse theorem! You might agree that Hecke's original idea and argument are remarkably beautiful. In 1949, in his seminal paper [68], Maass, a student of Hecke, extended Hecke's

method to non-holomorphic forms for $SL(2,\mathbb{Z})$. In 1967, the next very important step was made by Weil in his paper [85] dedicated to C. L. Siegel (1896-1981) celebrating Siegel's seventieth birthday. Weil showed how to work with Dirichlet series attached to holomorphic modular forms with respect to congruence subgroups of $SL(2,\mathbb{Z})$. He proved that if a Dirichlet series together with a sufficient number of twists satisfies *nice* analytic properties and functional equations with reasonably suitable regularity, then it stems from a holomorphic modular form with respect to a congruence subgroup of $SL(2,\mathbb{Z})$. More precisely his converse theorem can be formulated as follows.

Theorem B(Weil [85], 1967)**.** *Fix two positive integers d and N. Suppose the Dirichlet series*

$$D(s) = \sum_{n\geq 1} \frac{a_n}{n^s}$$

satisfies the following properties:
$(W1)$ *$D(s)$ converges absolutely for sufficiently large* $\mathrm{Re}(s) \gg 0$;
$(W2)$ *For every primitive character χ of modulus r with $(r,N)=1$, the function*

$$\Lambda(s,\chi) := (2\pi)^{-s}\,\Gamma(s) \sum_{n\geq 1} \frac{a_n\chi(n)}{n^s}$$

has an analytic continuation to an entire function of s to the whole complex plane, and is bounded in vertical strips of finite width;
$(W3)$ *Every such a function $\Lambda(s,\chi)$ satisfies the functional equation*

$$\Lambda(s,\chi) = w_\chi\, r^{-1}\,(r^2N)^{\frac{d}{2}-s}\,\Lambda(d-s,\overline{\chi}),$$

where

$$w_\chi = i^d\,\chi(N)\,g(\chi)^2$$

and

$$g(\chi) = \sum_{n\,(\mathrm{mod}\ r)} \chi(n)\,e^{2\pi i n/r}.$$

Then the function

$$f(\tau) = \sum_{n\geq 1} a_n e^{2\pi i n\tau}, \qquad \tau\in\mathbb{H} = \{\tau\in\mathbb{C}\mid \mathrm{Im}\,\tau > 0\}$$

is a holomorphic cusp form of weight d with respect to the congruence subgroup $\Gamma_0(N)$, where

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \,\middle|\, c\equiv 0 \pmod{N} \right\}.$$

Weil's proof follows closely Hecke's idea and argument. We see that his converse theorem provides a condition for modularity of the Dirichlet series $D(s)$ under $\Gamma_0(N)$ in terms of the functional equations of Dirichlet series *twisted by primitive characters*. In fact, Weil's converse theorem influenced the complete proof of the Shimura-Taniyama conjecture given by Wiles [86], Taylor et al. So we can say that the work of Weil marks the beginning of the modern era in the study of the connection between L-functions and automorphic forms.

In 1970 Jacquet and Langlands [33] established the converse theorem for $GL(2)$ in the adelic context of automorphic representations of $GL(2,\mathbb{A})$ based on Hecke's original idea. In 1979 Jacquet, Piatetski-Shapiro and Shalika [34] established the converse theorem for $GL(3)$ in the adelic context. Finally in 1994, generalizing the work on the converse theorems on $GL(2)$ and $GL(3)$, Cogdell and Piatetski-Shapiro [12], [15], [16] proved the converse theorem for $GL(n)$ with arbitrary $n \geq 1$ in the context of automorphic representations. The idea and technique in the proof of Cogdell and Piatetski-Shapiro are surprisingly almost the same as Hecke's. We now describe the converse theorems formulated and proved by them.

Let k be a global field, $\mathbb{A}$ its adele ring, and let ψ be a fixed nontrivial continuous additive character of $\mathbb{A}$ which is trivial on k. Let $\pi = \otimes_v \pi_v$ be an irreducible admissible representation of $GL(n,\mathbb{A})$, and let $\tau = \otimes_v \tau_v$ be a cuspidal automorphic representation of $GL(m,\mathbb{A})$ with $m < n$. We define formally

$$L(s,\pi\times\tau) = \prod_v L(s,\pi_v\times\tau_v) \quad \text{and} \quad \varepsilon(s,\pi\times\tau) = \prod_v \varepsilon(s,\pi_v\times\tau_v,\psi_v).$$

We say that $L(s,\pi\times\tau)$ is *nice* if it satisfies the following properties:
(N1) $L(s,\pi\times\tau)$ and $L(s,\widetilde{\pi}\times\widetilde{\tau})$ have analytic continuations to entire functions, where $\widetilde{\pi}$ (resp. $\widetilde{\tau}$) denotes the contragredient of π (resp. τ);
(N2) $L(s,\pi\times\tau)$ and $L(s,\widetilde{\pi}\times\widetilde{\tau})$ are bounded in vertical strips of finite width;
(N3) These entire functions satisfy the standard functional equation

$$L(s,\pi\times\tau) = \varepsilon(s,\pi\times\tau)\, L(1-s,\widetilde{\pi}\times\widetilde{\tau}).$$

Theorem C(Cogdell and Piatetski [15], [16], 1994)**.** *Let π be an irreducible admissible representation of $GL(n,\mathbb{A})$ whose central character is trivial on k^* and whose L-function $L(s,\pi)$ converges absolutely in some half plane. Assume that $L(s,\pi\times\tau)$ is nice for every cuspidal automorphic representation τ of $GL(m,\mathbb{A})$ for $1 \leq m \leq n-2$. Then π is a cuspidal automorphic repesentation of $GL(n,\mathbb{A})$.*

Furthermore they proved the following theorem.

Theorem D(Cogdell and Piatetski [16], 1999)**.** *Let π be an irreducible admissible representation of $GL(n,\mathbb{A})$ whose central character is trivial on k^* and whose L-function $L(s,\pi)$ converges absolutely in some half plane. Let S be a finite set of finite places. Assume that $L(s,\pi\times\tau)$ is nice for every cuspidal automorphic representation τ of $GL(m,\mathbb{A})$ for $1 \leq m \leq n-2$, which is unramified at the places in S. Then π*

is quasi-automorphic in the sense that there is an automorphic representation π' of $GL(n,\mathbb{A})$ such that $\pi_v \cong \pi'_v$ for all $v \notin S$.

The local converse theorem for $GL(n)$ was first formulated by Piatetski-Shapiro in his unpublished Maryland notes (1976) with his idea of deducing the local converse theorem from his global converse theorem. It was proved by Henniart [30] using a local approach. The local converse theorem is a basic ingredient in the proof of [LCC] for $GL(n)$ by Harris and Taylor [28] and by Henniart [31].

The local converse theorem for $GL(n)$ can be formulated as follows.

Theorem E(Henniart [30], 1993)**.** *Let k be a nonarchimedean local field of characteristic 0. Let τ and τ' be irreducible admissible generic representations of $GL(n,k)$ with the same central character. Assume the twisted local gamma factors (cf. [35]) are the same, i.e.,*

$$\gamma(s,\tau\times\rho,\psi)=\gamma(s,\tau'\times\rho,\psi)$$

for all irreducible supercuspidal representations ρ of $GL(m,k)$ with $1\le m\le n-1$. Then τ is isomorphic to τ'.

Remark 1. It is known that the twisting condition on m reduces from $n-1$ to $n-2$. It is expected as a conjecture of H. Jacquet [16, Conjecture 8.1] that the twisting condition on m should be reduced from $n-1$ to $[\frac{n}{2}]$.

Remark 2. The local converse theorem for generic representations of $U(2,1)$ and for $GSp(4)$ was established by E. M. Baruch in his Ph. D. thesis (Yale Univ., 1995).

Jiang and Soudry [38] proved the local converse theorem for irreducible admissible generic representations of $SO(2n+1,k)$.

Theorem F(Jiang and Soudry [38], 2003)**.** *Let σ and σ' be irreducible admissible generic representations of $SO(2n+1,k)$. Assume the twisted local gamma factors are the same, i.e.*

$$\gamma(s,\sigma\times\rho,\psi)=\gamma(s,\sigma'\times\rho,\psi)$$

for all irreducible supercuspidal representations ρ of $GL(m,k)$ with $1\le m\le 2n-1$. Then σ is isomorphic to σ'.

I shall give a brief sketch of the idea of their proof. They first reduce the proof of Theorem F to the case where both σ and σ' are supercuspidal by studying the existence of poles of twisted local gamma factors and related properties. Developing the explicit local Howe duality for irreducible admissible generic representations of $SO(2n+1,k)$ and the metaplectic group $\widetilde{Sp}(2n,k)$, and using the global weak Langlands functorial lifting form $SO(2n+1)$ to $GL(2n)$ (cf. Example 5.4, [13], [14]) and the local backward lifting from $GL(2n,k)$ to $\widetilde{Sp}(2n,k)$, they relate the local converse theorem for $SO(2n+1)$ with that for $GL(2n)$ which is well known now.

As an application of Theorem F, I repeat again that Jiang and Soudry [38], [39] proved the Local Langlands Reciprocity Law for $SO(2n+1)$. More precisely, there exists a *unique* bijective correspondence between the set of conjugacy classes of all

$2n$-dimensional, admissible, completely reducible, multiplicity-free, symplectic complex representations of the Weil group W_k and the set of all equivalence classes of irreducible generic supercuspidal representations of $SO(2n+1,k)$, which preserves the relevant local factors. As an application of Theorem F to the global theory, they proved that the weak Langlands functorial lifting from irreducible *generic* cuspidal automorphic representations of $SO(2n+1)$ to irreducible automorphic representations of $GL(2n)$ is *injective* up to isomorphism. It is still an open problem to establish the Langlands functorial lift from irreducible *non-generic* cuspidal automorphic representations of $SO(2n+1)$ to $GL(2n)$. As another application to the global theory, they proved the *rigidity theorem* in the sense that if $\pi = \otimes_v \pi_v$ and $\tau = \otimes_v \tau_v$ are irreducible generic cuspidal automorphic representations of $SO(2n+1,\mathbb{A})$ such that π_v is isomorphic to τ_v for almost all local places v, then π is isomorphic to τ.

References

[1] J. Adams, D. Barbasch abd D. Vogan, The Langlands Classification and Irreducible Characters for real Reductive Groups, Progr. Math. 104, Birkhäuser, Boston, 1992.

[2] J. Adams and D. Vogan, *L-groups, projective representations, and the Langlands classification*, Amer. J. Math., **114**(1992), 45-138.

[3] J. Arthur, *The principle of functoriality*, Bull. Amer. Math. Soc., **40(1)**(2003), 39-53.

[4] J. Arthur, *An Introduction to the Trace Formula, In Harmonic Analysis, the Trace Formula, and Shimura Varieties*, Proceedings of the Clay Mathematics Institute, 2003 Summer School, The Fields institute, Toronto, Canada, June 2-27, 2003, edited by J. Arthur, D. Ellwood and R. Kottwitz, Amer. Math. Soc. and CMI, 2005, 1-263.

[5] J. Arthur, *A notes on L-packets*, Pure and Applied Mathematics Quarterly, **2**(2006), 199-217.

[6] J. Arthur and L. Clozel, Simple Algebras, Base Change, and the Advanced Theory of the Trace Formula, Ann. of Math. Studies 120, Princeton Univ. Press, N.J., 1989.

[7] M. Asgari and F. Shahidi, *Generic transfer for general spin groups*, Duke Math. J., **132**(2006), 137-190.

[8] M. Asgari and F. Shahidi, *Generic transfer for GSp*(4) *to GL*(4), Compositio Math., **142**(2006), 541-550.

[9] A. Borel, *Automorphic L-functions*, Proc. Sympos. Pure Math., **33**, Part 2, Amer. Math. Soc., 1979, 27-62.

[10] K. Buzzard, M. Dickinson, N. Shepard-Barron and R. Taylor, *On Icosaheral Artin representations,* Duke Math. J., **109**(2001), 283-318.

[11] L. Clozel, M. Harris and R. Taylor, *Automorphy for some ℓ-adic lifts of automorphic mod ℓ representations* (preprint), http://www.math.harvard.edu/∼rtaylor/.

[12] J. Cogdell, Lectures on L-functions, Converse Theorems, and Functoriality for $GL(n)$, in Lectures in Automorphic L-functions, Fields institute Monographs, Amer. Math. Soc., 2004, 3-96.

[13] J. Cogdell, H. Kim, I. Piatetski-Shapiro, and F. Shahidi, *On lifting from classical groups to GL_N*, Publ. Math. Inst. Hautes Études Sci., **93**(2001), 5-30.

[14] J. Cogdell, H. Kim, I. Piatetski-Shapiro, and F. Shahidi, *Functoriality for the classical groups*, Publ. Math. Inst. Hautes Études Sci., **99**(2004), 163-233.

[15] J. Cogdell and I. Piatetski-Shapiro, *Converse theorems for GL_n*, Publ. Math. Inst. Hautes Études Sci., **79**(1994), 157-204.

[16] J. Cogdell and I. Piatetski-Shapiro, *Converse theorems for GL_n, II*, J. reine angew. Math., **507**(1999), 165-188.

[17] P. Deligne, Formes modulaires et représentations ℓ-adiques: Sém. Bourbaki 1968/9, no. 355; Lecture Notes in Math. 179, Springer-Verlag, Berlin-Heidelberg-New York, 1971, 139-172.

[18] P. Deligne, Formes modulaires et représentations of $GL(2)$, Antwerp II, in Modular Forms of One Variable II, Lecture Notes in Math. 349, Springer-Verlag, New York, 1973, 55-106.

[19] P. Deligne, Les constants des équation fontionnelle des fonctions L , in Modular Forms of One Variable II, Lecture Notes in Math., 349, Springer-Verlag, New York, 1973, 501-595.

[20] V. Drinfeld, *Proof of the Petersson conjecture for $GL(2)$ over a global field of characteristic p*, Funct. Anal. and its Appl., **22**(1988), 28-43.

[21] V. Drinfeld, *Cohomology of compactified modules of F-sheaves of rank* 2, J. Soviet Math., **46**(1989), 1789-1821.

[22] S. Gelbart and H. Jacquet, *A relation between automorphic representation of $GL(2)$ and $GL(3)$*, Ann. Scient. Éc. Normale Sup., **11**(1978), 471-552.

[23] S. Gelbart and F. Shahidi, *Analytic Properties of Automorphic L-Functions*, Perspective in Mathematics, **6**(1988), Academic Press.

[24] S. Gelbart, I. Piatetski-Shapiro and S. Rallis, Explicit Constructions of Automorphic L-Functions, Lecture Notes in Math. 1254, Springer-Verlag, New York, 1987.

[25] D. Ginzburg, S. Rallis and D. Soudry, *Generic automorphic forms on $SO(2n+1)$: functorial lift on $GL(2n)$, endoscopy, and the base change*, Internat. Math. Res. Notices, no. **14**(2001), 729-764.

[26] H. Hamburger, *Über die Riemannsche Funktionalgleichung der ζ-Funktion*, Math. Zeit., **10**(1921), 240-254; **11**(1922), 224-245; **13**(1922), 283-311.

[27] M. Harris, N. Shepherd-Barron and R. Taylor, *Ihara's lemma and potential automorphy* (preprint), http://www.math.harvard.edu/∼rtaylor/.

[28] M. Harris and R. Taylor, On the Geonmetry and Cohomology of Some Simple Shimura Varieties, Ann. of Math. Studies 151, Princeton Univ. Press, Princeton, N. J., 2001.

[29] E. Hecke, *Über die Bestimmung Dirichletscher Reihnen durch Funktionalgleichungen*, Math. Ann., **112**(1936), 664-699.

[30] G. Henniart, *Caractérisation de la correspondance de Langlands locale par les facteurs ε de paires*, Invent. Math., **113**(1993), 339-350.

[31] G. Henniart, *Une preuve simple des conjectures de Langlands de $GL(n)$ sur un corps p-adique*, Invent. Math., **139**(2000), 439-455.

[32] T. Ikeda, *On the location of poles of the triple L-functions*, Compositio Math., **83**(1992), 187-237.

[33] H. Jacquet and R. Langlands, Automorphic Forms on $GL(2)$, Lecture Notes in Math. 114, Springer-Verlag, New York, 1970.

[34] H. Jacquet, I. Piatetski-Shapiro and J. Shalika, *Automorphic forms on* $GL(3)$, I and II, Ann. of Math., **109**(1979), 169-258.

[35] H. Jacquet, I. Piatetski-Shapiro and J. Shalika, *Rankin-Selberg convolutions*, Amer. J. Math., **105**(1983), 367-464.

[36] H. Jacquet and J. Shalika, *On Euler products and the classification of automorphic representations* I, Amer. J. Math., **103**(1981), 499-558.

[37] H. Jacquet and J. Shalika, *On Euler products and the classification of automorphic representations* II, Amer. J. Math., **103**(1981), 777-815.

[38] D. Jiang and D. Soudry, *The local converse theorem for* $SO(2n+1)$ *and applications*, Ann. Math., **157**(2003), 743-806.

[39] D. Jiang and D. Soudry, Generic representations and local langlands reciprocity law for p-adic SO_{2n+1}, in Contributions to Automorphic Forms, Geometry and Number Theory, edited by H. Hida, D. Ramakrishnan and F. Shahidi, the John Hopkins University Press, Baltimore, 2004, 457-519.

[40] D. Kazhdan and G. Lusztig, *Proof of the Deligne-Langlands conjecture for Hecke algebras*, Invent. Math., **95**(1987), 153-215.

[41] C. Khare, M. Larsen and G. Savin, *Functoriality and the inverse Galois problem*, Compositio Math., **144**(2008), 541-564.

[42] C. Khare, M. Larsen and G. Savin, *Functoriality and the inverse Galois problem II: Groups of type* B_n *and* G_2, arXiv:0807.0861v1, 2008.

[43] H. Kim, *Functoriality for the exterior square of* GL_4 *and the symmetric fourth of* GL_2, J. of Amer. Math. Soc., **16(1)**(2002), 139-183.

[44] H. Kim, Automorphic L-functions, in Lectures in Automorphic L-functions, Fields institute Monographs, Amer. Math. Soc., 2004, 97-201.

[45] H. Kim and F. Shahidi, *Symmetric cube L-functions for* GL_2 *are entire*, Ann. of Math., **150**(1999), 645-662.

[46] H. Kim and F. Shahidi, *Functional products for* $GL_2 \times GL_3$ *and functorial symmetric cube for* GL_2, Ann. of Math., **155**(2002), 837-893.

[47] H. Kim and F. Shahidi, *Cuspidality of symmetric powers with applications*, Duke Math. J., **112**(2002), 177-197.

[48] A. Knapp and G. Zuckerman, Normalizing factors and L-groups, Proc. Symp. Pure Math., Amer. Math. Soc., Providence, RI, 33, Part 1, 1979, 93-106.

[49] A. Knapp and G. Zuckerman, *Classification of irreducible tempered representations of semisimple groups*, Ann. of Math., **116**(1982), 389-455.

[50] R. Kottwitz and D. Shelstad, *Foundations of Twisted Endoscopy*, Astérisque, **255**(1999).

[51] J. -P. Labesse and R. Langlands, *L-indistinguishabilty for* $SL(2)$, Canad. J. Math., **31**(1979), 726-785.

[52] L. Lafforgue, *Chtoucas de Drinfeld et correspondance de Langlands*, Invent. Math., **147**(2002), 1-241.

[53] K. F. Lai, On the Tamagawa number of quasi-split groups, Thesis, Yale University, 1974.

[54] R. Langlands, Problems in the theory of automorphic forms, in Lectures in modern analysis and Applications, Lecture Notes in Math. 170, Springer-Verlag, New York, 1970, 18-61.

[55] R. Langlands, Euler Products, Yale University Press, New Haven 310, 1971.

[56] R. Langlands, Modular forms and ℓ-adic representations, in Modular Forms of One Variable II, Lecture Notes in Math. 349, Springer-Verlag, New York, 1973, 361-500.

[57] R. Langlands, On Artin's L-functions, Rice University Studies 56, 23-28.

[58] R. Langlands, On the Functional Equations Satisfied by Eisenstein Series, Lecture Notes in Math. 544, Springer-Verlag, New York, 1976.

[59] R. Langlands, *On the notion of automorphic representations. A supplement to the preceding paper*, Proc. Sympos. Pure Math. **33**, Part 1, Amer. Math. Soc., 1979, 203-208.

[60] R. Langlands, *Automorphic representations, Shimura varieties, and motives*, Proc. Sympos. Pure Math., **33**, Part 2, Amer. Math. Soc., 1979, 205-246.

[61] R. Langlands, Base Change for $GL(2)$, Ann. of Math. Studies 96, Princeton Univ. Press, Princeton, N. J. 1980.

[62] R. Langlands, *Les débuts d'une formule des traces stables*, Publ. Math. Ubiv. Paris VII., **13**(1983).

[63] R. Langlands, *On the classiciation of irreducible representations of real algebraic groups*, in *Representation Theory and Harmonic Analysis on Semisimple Lie Groups*, AMS Mathematical Surveys and Monographs, **31**(1989), 101-170.

[64] R. Langlands, *Representations of abelian algebraic groups*, Pacific J. Math., 1997, 231-250.

[65] R. Langlands, Where stands functoriality today?, in Representation Theory and Automorphic Forms (March 1996, Edinburgh, Scotland), edited by T. N. Bailey and A. W. Knapp, 1997, 457-471.

[66] R. Langlands, Beyond endoscopy, in Contributions to Automorphic Forms, Geometry and Number Theory, edited by H. Hida, D. Ramakrishnan and F. Shahidi, the John Hopkins University Press, Baltimore 2004, 611-698.

[67] G. Laumon, M. Rapoport and U. Stuhler, *$\mathcal{D}$-eliptic sheaves and the Langlands correspondence*, Invent. Math., **113**(1993), 217-338.

[68] H. Maass, *Über eine neue Art von nichtanalytischen automorphen Funktionen und die Bestimmung Dirichletscher Reihnen durch Funktionalgleichungen*, Math. Ann., **121**(1949), 141-183.

[69] B. Mazur, *Finding Meaning in Error Terms*, Bull. Amer. Math. Soc., **45(2)**April(2008), 185-228.

[70] D. Ramakrishnan, *Pure motives and automorphic forms*, in Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., **55**, Part 1, Amer. Math. Soc., 1994, 377-400.

[71] D. Ramakrishnan, *Modularity of the Rankin-Selberg L-series, and multiplicity one for* $SL(2)$, Ann. of Math., **152**(2000), 45-111.

[72] D. Ramakrishnan, Modularity of solvable Artin representations of $GO(4)$-type, Int. Math. Research Notices, no. 1 (2002), 1-54.

[73] J.-P. Serre, Abelian ℓ-adic Representations and Elliptic Curves, Benjamin, New York, 1968.

[74] J.-P. Serre, *Propriétés conjecturales des groupes de Galois motiviques et des representations ℓ-adiques*, in Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math. **55**, Part 1, Amer. Math. Soc., 1994, 377-400.

[75] F. Shahidi, *Functional equations satisfied by certain L-functions*, Compositio Math., **37**(1978), 171-208.

[76] F. Shahidi, *On certain L-functions*, Amer. J. Math., **103**(1981), 297-356.

[77] F. Shahidi, *On the Ramanujan conjecture and finiteness of poles for certain L-functions*, Ann. of Math., **127**(1988), 547-584.

[78] F. Shahidi, *A proof of Langlands' conjecture on Plancherel measures; Complementary series for p-adic groups*, Ann. of Math., **132**(1988), 273-330.

[79] F. Shahidi, *Local coefficients as Mellin transforms of Bessel functions; towards a general stability*, IMRN, **39**(2002), 2075-2119.

[80] F. Shahidi, *Third symmetric powe L-functions for* $GL(2)$, Compositio Math., **70**(1989), 245-273.

[81] F. Shahidi, *Langlands Functoriality Conjecture and Number Theory, In Representation Theory and Automorphic Forms*, edited by T. Kobayashi, W. Schmid and J. -H. Yang, Progress in Mathematics, Birkhäser, **255**(2008), 151-173.

[82] J. Tate, *Number theoretic background, in Automorphic Forms, representation Theory and L-functions*, Proc. Sympos. Pure Math., **33**, Part 2, Amer. Math. Soc., 1979, 3-26.

[83] R. Taylor, *Automorphy for some ℓ-adic lifts of automorphic mod ℓ representations* II (preprint), http://www.math.harvard.edu/∼rtaylor/.

[84] J. Tunnell, *Artin's conjecture for representations of octahedral type,* Bull. Amer. Math. Soc., **5**(1981), 173-175.

[85] A. Weil, *Über die Bestimmung Dirichletscher Reihnen durch Funktionalgleichungen*, Math. Ann., **168**(1967), 149-156.

[86] A. Wiles, *Modular elliptic curves and Fermat's Last Theorem*, Ann. of Math., **141**(1995), 443-551.