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Lattice representations of Heisenberg groups

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1. Introduction

For any positive integers g and h, we consider the Heisenberg group

$$H_{\mathbb{R}}^{(g,h)} := \left\{ (\lambda, \mu, \kappa) \mid \lambda, \mu \in \mathbb{R}^{(h,g)},\ \kappa \in \mathbb{R}^{(h,h)},\ \kappa + \mu\, {}^t\lambda \text{ symmetric} \right\}.$$

Recall that the multiplication law is

$$(\lambda, \mu, \kappa) \circ (\lambda', \mu', \kappa') := (\lambda + \lambda', \mu + \mu', \kappa + \kappa' + \lambda\, {}^t\mu' - \mu\, {}^t\lambda').$$

Here $\mathbb{R}^{(h,g)}$ (resp. $\mathbb{R}^{(h,h)}$) denotes the set of all $h \times g$ (resp. $h \times h$) real matrices.

The Heisenberg group $H_{\mathbb{R}}^{(g,h)}$ is embedded into the symplectic group $Sp(g+h, \mathbb{R})$ via the mapping

$$H_{\mathbb{R}}^{(g,h)} \ni (\lambda, \mu, \kappa) \longmapsto \begin{pmatrix} E_g & 0 & 0 & {}^t\mu \\ \lambda & E_h & \mu & \kappa \\ 0 & 0 & E_g & -{}^t\lambda \\ 0 & 0 & 0 & E_h \end{pmatrix} \in Sp(g+h, \mathbb{R}).$$

This Heisenberg group is a 2-step nilpotent Lie group and is important in the study of toroidal compactifications of Siegel moduli spaces. In fact, $H_{\mathbb{R}}^{(g,h)}$ is obtained as the unipotent radical of the parabolic subgroup of $Sp(g+h, \mathbb{R})$ associated with the rational boundary component F_g (cf. [F-C] p. 123 or [N] p. 21). For the motivation of the study of this Heisenberg group we refer to [Y4]-[Y8] and [Z]. We refer to [Y1]-[Y3] for more results on $H_{\mathbb{R}}^{(g,h)}$.

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In [C], P. Cartier stated without proof that for $h = 1$, the lattice representation of $H_{\mathbb{R}}^{(g,1)}$ associated to the lattice L is unitarily equivalent to the direct sum of $[L^* : L]^{\frac{1}{2}}$ copies of the Schrödinger representation of $H_{\mathbb{R}}^{(g,1)}$, where L^* is the dual lattice of L with respect to a certain nondegenerate alternating bilinear form. R. Berndt proved the above fact for the case $h = 1$ in his lecture notes [B]. In this paper, we give a complete proof of Cartier's theorem for $H_{\mathbb{R}}^{(g,h)}$.

Main Theorem. Let $\mathcal{M}$ be a positive definite, symmetric half-integral matrix of degree h and L be a self-dual lattice in $\mathbb{C}^{(h,g)}$. Then the lattice representation $\pi_{\mathcal{M}}$ of $H_{\mathbb{R}}^{(g,h)}$ associated with L and $\mathcal{M}$ is unitarily equivalent to the direct sum of $(\det 2\mathcal{M})^g$ copies of the Schrödinger representation of $H_{\mathbb{R}}^{(g,h)}$. For more details, we refer to Sect. 3.

The paper is organized as follows. In Sect. 2, we review the Schrödinger representations of the Heisenberg group $H_{\mathbb{R}}^{(g,h)}$. In Sect. 3, we prove the main theorem. In the final section, we provide a relation between lattice representations and theta functions.

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Notations. We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of integers, the field of real numbers, and the field of complex numbers respectively. The symbol $\mathbb{C}_1^\times$ denotes the multiplicative group consisting of all complex numbers z with $|z| = 1$, and the symbol $Sp(g, \mathbb{R})$ the symplectic group of degree g, H_g the Siegel upper half plane of degree g. The symbol ":=" means that the expression on the right hand side is the definition of that on the left. We denote by $\mathbb{Z}^+$ the set of all positive integers, by $F^{(k,l)}$ the set of all $k \times l$ matrices with entries in a commutative ring F. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transpose matrix of M. For $A \in F^{(k,k)}$, $\sigma(A)$ denotes the trace of A. For $A \in F^{(k,l)}$ and $B \in F^{(k,k)}$, we set $B[A] = {}^tABA$. We denote the identity matrix of degree k by E_k. For a positive integer n, $\mathrm{Symm}\,(n, K)$ denotes the vector space consisting of all symmetric $n \times n$ matrices with entries in a field K.

2. Schrödinger representations

First of all, we observe that $H_{\mathbb{R}}^{(g,h)}$ is a 2-step nilpotent Lie group. It is easy to see that the inverse of an element $(\lambda, \mu, \kappa) \in H_{\mathbb{R}}^{(g,h)}$ is given by

$$(\lambda, \mu, \kappa)^{-1} = (-\lambda, -\mu, -\kappa + \lambda\, {}^t\mu - \mu\, {}^t\lambda).$$

Now we set

$$(2.1)\qquad [\lambda,\mu,\kappa] := (0,\mu,\kappa)\circ(\lambda,0,0) = (\lambda,\mu,\kappa-\mu\,{}^t\lambda).$$

Then $H_{\mathbb{R}}^{(g,h)}$ may be regarded as a group equipped with the following multiplication

$$(2.2)\qquad [\lambda,\mu,\kappa]\diamond[\lambda_0,\mu_0,\kappa_0] := [\lambda+\lambda_0,\mu+\mu_0,\kappa+\kappa_0+\lambda\,{}^t\mu_0+\mu_0\,{}^t\lambda].$$

The inverse of $[\lambda,\mu,\kappa]\in H_{\mathbb{R}}^{(g,h)}$ is given by

$$[\lambda,\mu,\kappa]^{-1} = [-\lambda,-\mu,-\kappa+\lambda\,{}^t\mu+\mu\,{}^t\lambda].$$

We set

$$(2.3)\qquad K := \left\{ [0,\mu,\kappa]\in H_{\mathbb{R}}^{(g,h)} \;\middle|\; \mu\in\mathbb{R}^{(h,g)},\ \kappa = {}^t\kappa\in\mathbb{R}^{(h,h)} \right\}.$$

Then K is a commutative normal subgroup of $H_{\mathbb{R}}^{(g,h)}$. Let $\hat{K}$ be the Pontrajagin dual of K, i.e., the commutative group consisting of all unitary characters of K. Then $\hat{K}$ is isomorphic to the additive group $\mathbb{R}^{(h,g)}\times \mathrm{Symm}\,(h,\mathbb{R})$ via

$$(2.4)\qquad < a,\hat{a} > := e^{2\pi i\sigma(\hat{\mu}\,{}^t\mu+\hat{\kappa}\kappa)},\quad a=[0,\mu,\kappa]\in K,\ \hat{a}=(\hat{\mu},\hat{\kappa})\in\hat{K}.$$

We put

$$(2.5)\qquad S := \left\{ [\lambda,0,0]\in H_{\mathbb{R}}^{(g,h)} \;\middle|\; \lambda\in\mathbb{R}^{(h,g)} \right\} \cong \mathbb{R}^{(h,g)}.$$

Then S acts on K as follows:

$$(2.6)\qquad \alpha_\lambda([0,\mu,\kappa]) := [0,\mu,\kappa+\lambda\,{}^t\mu+\mu\,{}^t\lambda],\quad [\lambda,0,0]\in S.$$

It is easy to see that the Heisenberg group $\left(H_{\mathbb{R}}^{(g,h)},\diamond\right)$ is isomorphic to the semi-direct product $S\ltimes K$ of S and K whose multiplication is given by

$$(\lambda,a)\cdot(\lambda_0,a_0) := (\lambda+\lambda_0, a+\alpha_\lambda(a_0)),\ \ \lambda,\lambda_0\in S,\ a,a_0\in K.$$

On the other hand, S acts on $\hat{K}$ by

$$(2.7)\qquad \alpha_\lambda^*(\hat{a}) := (\hat{\mu}+2\hat{\kappa}\lambda,\hat{\kappa}),\ \ [\lambda,0,0]\in S,\ \ a=(\hat{\mu},\hat{\kappa})\in\hat{K}.$$

Then, we have the relation $<\alpha_\lambda(a),\hat{a}> = < a,\alpha_\lambda^*(\hat{a})>$ for all $a\in K$ and $\hat{a}\in\hat{K}$.

We have two types of S-orbits in $\hat{K}$.

TYPE I. Let $\hat{\kappa}\in\mathrm{Symm}\,(h,\mathbb{R})$ with $\hat{\kappa}\neq 0$. The S-orbit of $\hat{a}(\hat{\kappa}) := (0,\hat{\kappa})\in\hat{K}$ is given by

$$(2.8)\qquad \hat{\mathcal{O}}_{\hat{\kappa}} := \left\{ (2\hat{\kappa}\lambda,\hat{\kappa})\in\hat{K} \;\middle|\; \lambda\in\mathbb{R}^{(h,g)} \right\} \cong \mathbb{R}^{(h,g)}.$$

TYPE II. Let $\hat{y} \in \mathbb{R}^{(h,g)}$. The S-orbit $\hat{\mathcal{O}}_{\hat{y}}$ of $\hat{a}(\hat{y}) := (\hat{y}, 0)$ is given by

$$\hat{\mathcal{O}}_{\hat{y}} := \left\{ (\hat{y}, 0) \right\} = \hat{a}(\hat{y}). \tag{2.9}$$

We have

$$\hat{K} = \left(\bigcup_{\hat{\kappa} \in \mathrm{Symm}(h,\mathbb{R})} \hat{\mathcal{O}}_{\hat{\kappa}} \right) \bigcup \left(\bigcup_{\hat{y} \in \mathbb{R}^{(h,g)}} \hat{\mathcal{O}}_{\hat{y}} \right)$$

as a set. The stabilizer $S_{\hat{\kappa}}$ of S at $\hat{a}(\hat{\kappa}) = (0, \hat{\kappa})$ is given by

$$S_{\hat{\kappa}} = \{0\}. \tag{2.10}$$

And the stabilizer $S_{\hat{y}}$ of S at $\hat{a}(\hat{y}) = (\hat{y}, 0)$ is given by

$$S_{\hat{y}} = \left\{ [\lambda, 0, 0] \,\middle|\, \lambda \in \mathbb{R}^{(h,g)} \right\} = S \cong \mathbb{R}^{(h,g)}. \tag{2.11}$$

From now on, we set $G := H_{\mathbb{R}}^{(g,h)}$ for brevity. It is known that K is a closed, commutative normal subgroup of G. Since $(\lambda, \mu, \kappa) = (0, \mu, \kappa + \mu\, {}^t\lambda) \circ (\lambda, 0, 0)$ for $(\lambda, \mu, \kappa) \in G$, the homogeneous space $X := K \backslash G$ can be identified with $\mathbb{R}^{(h,g)}$ via

$$Kg = K \circ (\lambda, 0, 0) \longmapsto \lambda, \quad g = (\lambda, \mu, \kappa) \in G.$$

We observe that G acts on X by

$$(Kg) \cdot g_0 := K\,(\lambda + \lambda_0, 0, 0) = \lambda + \lambda_0, \tag{2.12}$$

where $g = (\lambda, \mu, \kappa) \in G$ and $g_0 = (\lambda_0, \mu_0, \kappa_0) \in G$.

If $g = (\lambda, \mu, \kappa) \in G$, we have

$$k_g = (0, \mu, \kappa + \mu\, {}^t\lambda), \quad s_g = (\lambda, 0, 0) \tag{2.13}$$

in the Mackey decomposition of $g = k_g \circ s_g$ (cf. [M]). Thus if $g_0 = (\lambda_0, \mu_0, \kappa_0) \in G$, then we have

$$s_g \circ g_0 = (\lambda, 0, 0) \circ (\lambda_0, \mu_0, \kappa_0) = (\lambda + \lambda_0, \mu_0, \kappa_0 + \lambda\, {}^t\mu_0) \tag{2.14}$$

and so

$$k_{s_g \circ g_0} = (0, \mu_0, \kappa_0 + \mu_0\, {}^t\lambda_0 + \lambda\, {}^t\mu_0 + \mu_0\, {}^t\lambda). \tag{2.15}$$

For a real symmetric matrix $c = {}^tc \in \mathbb{R}^{(h,h)}$ with $c \neq 0$, we consider the one-dimensional unitary representation σ_c of K defined by

$$\sigma_c\,((0, \mu, \kappa)) := e^{2\pi i \sigma(c\kappa)}\, I, \quad (0, \mu, \kappa) \in K, \tag{2.16}$$

where I denotes the identity mapping. Then the induced representation $U(\sigma_c) := \mathrm{Ind}_K^G\, \sigma_c$ of G induced from σ_c is realized in the Hilbert space $\mathcal{H}_{\sigma_c} = L^2(X, d\dot{g},$

$\mathbb{C}) \cong L^2\left(\mathbb{R}^{(h,g)}, d\xi\right)$ as follows. If $g_0 = (\lambda_0, \mu_0, \kappa_0) \in G$ and $x = Kg \in X$ with $g = (\lambda, \mu, \kappa) \in G$, we have

$$(2.17)\qquad \left(U_{g_0}(\sigma_c) f\right)(x) = \sigma_c\left(k_{s_g \circ g_0}\right)(f(xg_0)), \quad f \in \mathcal{H}_{\sigma_c}.$$

It follows from (2.15) that

$$(2.18)\qquad \left(U_{g_0}(\sigma_c) f\right)(\lambda) = e^{2\pi i \sigma\{c(\kappa_0 + \mu_0\, {}^t\lambda_0 + 2\lambda\, {}^t\mu_0)\}} f(\lambda + \lambda_0).$$

Here, we identified $x = Kg$ (resp. $xg_0 = Kgg_0$) with λ (resp. $\lambda + \lambda_0$). The induced representation $U(\sigma_c)$ is called the *Schrödinger representation* of G associated with σ_c. Thus $U(\sigma_c)$ is a monomial representation.

Now, we denote by $\mathcal{H}^{\sigma_c}$ the Hilbert space consisting of all functions $\phi : G \longrightarrow \mathbb{C}$ which satisfy the following conditions:

(1) $\phi(g)$ is measurable with respect to dg,
(2) $\phi((0, \mu, \kappa) \circ g)) = e^{2\pi i \sigma(c\kappa)} \phi(g)$ for all $g \in G$,
(3) $\| \phi \|^2 := \int_X |\phi(g)|^2 d\dot{g} < \infty, \quad \dot{g} = Kg$,

where dg (resp. $d\dot{g}$) is a G-invariant measure on G (resp. $X = K \backslash G$). The inner product $(\ ,\ )$ on $\mathcal{H}^{\sigma_c}$ is given by

$$(\phi_1, \phi_2) := \int_G \phi_1(g)\, \overline{\phi_2(g)}\, dg \quad \text{for } \phi_1,\ \phi_2 \in \mathcal{H}^{\sigma_c}.$$

We observe that the mapping $\Phi_c : \mathcal{H}_{\sigma_c} \longrightarrow \mathcal{H}^{\sigma_c}$ defined by

$$(2.19)\qquad (\Phi_c(f))(g) := e^{2\pi i \sigma\{c(\kappa + \mu\, {}^t\lambda)\}} f(\lambda), \quad f \in \mathcal{H}_{\sigma_c},\ g = (\lambda, \mu, \kappa) \in G$$

is an isomorphism of Hilbert spaces. The inverse $\Psi_c : \mathcal{H}^{\sigma_c} \longrightarrow \mathcal{H}_{\sigma_c}$ of Φ_c is given by

$$(2.20)\qquad (\Psi_c(\phi))(\lambda) := \phi((\lambda, 0, 0)), \quad \phi \in \mathcal{H}^{\sigma_c},\ \lambda \in \mathbb{R}^{(h,g)}.$$

The Schrödinger representation $U(\sigma_c)$ of G on $\mathcal{H}^{\sigma_c}$ is given by

$$(2.21)\qquad \left(U_{g_0}(\sigma_c)\phi\right)(g) = e^{2\pi i \sigma\{c(\kappa_0 + \mu_0\, {}^t\lambda_0 + \lambda\, {}^t\mu_0 - \lambda_0\, {}^t\mu)\}} \phi\left((\lambda_0, 0, 0) \circ g\right),$$

where $g_0 = (\lambda_0, \mu_0, \kappa_0),\ g = (\lambda, \mu, \kappa) \in G$ and $\phi \in \mathcal{H}^{\sigma_c}$. (2.21) can be expressed as follows.

$$(2.22)\qquad \left(U_{g_0}(\sigma_c)\phi\right)(g) = e^{2\pi i \sigma\{c(\kappa_0 + \kappa + \mu_0\, {}^t\lambda_0 + \mu\, {}^t\lambda + 2\lambda\, {}^t\mu_0)\}} \phi((\lambda_0 + \lambda, 0, 0)).$$

Theorem 2.1. Let c be a positive symmetric half-integral matrix of degree h. Then the Schrödinger representation $U(\sigma_c)$ of G is irreducible.

Proof. The proof can be found in [Y1], theorem 3. □

3. Proof of the Main Theorem

Let $L := \mathbb{Z}^{(h,g)} \times \mathbb{Z}^{(h,g)}$ be the lattice in the vector space $V \cong \mathbb{C}^{(h,g)}$. Let B be an alternating bilinear form on V such that $B(L, L) \subset \mathbb{Z}$, that is, $\mathbb{Z}$-valued on $L \times L$. The dual L_B^* of L with respect to B is defined by

$$L_B^* := \{\, v \in V \mid B(v, l) \in \mathbb{Z} \text{ for all } l \in L \,\}.$$

Then $L \subset L_B^*$. If B is nondegenerate, L_B^* is also a lattice in V, called the *dual lattice* of L. In case B is nondegenerate, there exist a $\mathbb{Z}$-basis $\{\, \xi_{11}, \xi_{12}, \cdots, \xi_{hg}, \eta_{11}, \eta_{12}, \cdots, \eta_{hg} \,\}$ of L and a set $\{\, e_{11}, e_{12}, \cdots, e_{hg} \,\}$ of positive integers such that $e_{11}|e_{12},\ e_{12}|e_{13}, \cdots, e_{h,g-1}|e_{hg}$ for which

$$\begin{pmatrix} B(\xi_{ka}, \xi_{lb}) & B(\xi_{ka}, \eta_{lb}) \\ B(\eta_{ka}, \xi_{lb}) & B(\eta_{ka}, \eta_{lb} \end{pmatrix} = \begin{pmatrix} 0 & e \\ -e & 0 \end{pmatrix},$$

where $1 \le k, l \le h,\ 1 \le a, b \le g$ and $e := \operatorname{diag}(e_{11}, e_{12}, \cdots, e_{hg})$ is the diagonal matrix of degree hg with entries $e_{11}, e_{12}, \cdots, e_{hg}$. It is well known that $[L_B^* : L] = (\det e)^2 = (e_{11}e_{12}\cdots e_{hg})^2$ (cf. [I] p. 72). The number $\det e$ is called the *Pfaffian* of B.

Now, we consider the following subgroups of G:

(3.1) $$\Gamma_L := \left\{ (\lambda, \mu, \kappa) \in G \mid (\lambda, \mu) \in L,\ \kappa \in \mathbb{R}^{(h,h)} \right\}$$

and

(3.2) $$\Gamma_{L_B^*} := \left\{ (\lambda, \mu, \kappa) \in G \mid (\lambda, \mu) \in L_B^*,\ \kappa \in \mathbb{R}^{(h,h)} \right\}.$$

Then both Γ_L and $\Gamma_{L_B^*}$ are normal subgroups of G. We set

(3.3) $$\mathcal{Z}_0 := \left\{ (0, 0, \kappa) \in G \mid \kappa = {}^t\kappa \in \mathbb{Z}^{(h,h)} \text{ integral} \right\}.$$

It is easy to show that

$$\Gamma_{L_B^*} = \left\{ g \in G \mid g\gamma g^{-1}\gamma^{-1} \in \mathcal{Z}_0 \text{ for all } \gamma \in \Gamma_L \right\}.$$

We define

(3.4) $$Y_L := \left\{ \varphi \in \operatorname{Hom}(\Gamma_L, \mathbb{C}_1^\times) \mid \varphi \text{ is trivial on } \mathcal{Z}_0 \right\}$$

and

(3.5) $$Y_{L,S} := \left\{ \varphi \in Y_L \mid \varphi(\kappa) = e^{2\pi i\sigma(S\kappa)} \text{ for all } \kappa = {}^t\kappa \in \mathbb{R}^{(h,h)} \right\}$$

for each symmetric real matrix S of degree h. We observe that, if S is not half-integral, then $Y_L = \emptyset$ and so $Y_{L,S} = \emptyset$. It is clear that, if S is symmetric half-integral, then $Y_{L,S}$ is not empty.

Thus we have

$$Y_L = \cup_{\mathcal{M}} Y_{L,\mathcal{M}}, \tag{3.6}$$

where $\mathcal{M}$ runs through the set of all symmetric half-integral matrices of degree h.

Lemma 3.1. Let $\mathcal{M}$ be a symmetric half-integral matrix of degree h with $\mathcal{M} \neq 0$. Then any element φ of $Y_{L,\mathcal{M}}$ is of the form $\varphi_{\mathcal{M},q}$. Here $\varphi_{\mathcal{M},q}$ is the character of Γ_L defined by

$$\varphi_{\mathcal{M},q}((l,\kappa)) := e^{2\pi i \sigma(\mathcal{M}\kappa)} \cdot e^{\pi i q(l)} \quad \text{for } (l,\kappa) \in \Gamma_L, \tag{3.7}$$

where $q : L \longrightarrow \mathbb{R}/2\mathbb{Z} \cong [0,2)$ is a function on L satisfying the following condition:

$$q(l_0 + l_1) \equiv q(l_0) + q(l_1) - 2\sigma\{\mathcal{M}(\lambda_0\, {}^t\mu_1 - \mu_0\, {}^t\lambda_1)\} \pmod 2 \tag{3.8}$$

for all $l_0 = (\lambda_0, \mu_0) \in L$ and $l_1 = (\lambda_1, \mu_1) \in L$.

Proof. (3.8) follows immediately from the fact that $\varphi_{\mathcal{M},q}$ is a character of Γ_L. It is obvious that any element of $Y_{L,\mathcal{M}}$ is of the form $\varphi_{\mathcal{M},q}$. □

Lemma 3.2. An element of $Y_{L,0}$ is of the form $\varphi_{k,l}$ $(k, l \in \mathbb{R}^{(h,g)})$. Here $\varphi_{k,l}$ is the character of Γ_L defined by

$$\varphi_{k,l}(\gamma) := e^{2\pi i \sigma(k\, {}^t\lambda + l\, {}^t\mu)}, \quad \gamma = (\lambda, \mu, \kappa) \in \Gamma_L. \tag{3.9}$$

Proof. It is easy to prove and so we omit the proof. □

Lemma 3.3. Let $\mathcal{M}$ be a nonsingular symmetric half-integral matrix of degree h. Let $\varphi_{\mathcal{M},q_1}$ and $\varphi_{\mathcal{M},q_2}$ be the characters of Γ_L defined by (3.7). The character φ of Γ_L defined by $\varphi := \varphi_{\mathcal{M},q_1} \cdot \varphi_{\mathcal{M},q_2}^{-1}$ is an element of $Y_{L,0}$.

Proof. It follows from the existence of an element $g = (\mathcal{M}^{-1}\lambda, \mathcal{M}^{-1}\mu, 0) \in G$ with $(\lambda, \mu) \in V$ such that

$$\varphi_{\mathcal{M},q_1}(\gamma) = \varphi_{\mathcal{M},q_2}(g\gamma g^{-1}) \quad \text{for all } \gamma \in \Gamma_L.$$

□

For a unitary character $\varphi_{\mathcal{M},q}$ of Γ_L defined by (3.7), we let

$$\pi_{\mathcal{M},q} := \operatorname{Ind}_{\Gamma_L}^G \varphi_{\mathcal{M},q} \tag{3.10}$$

be the representation of G induced from $\varphi_{\mathcal{M},q}$. Let $\mathcal{H}_{\mathcal{M},q}$ be the Hilbert space consisting of all measurable functions $\phi : G \longrightarrow \mathbb{C}$ satisfying

(L1) $\phi(\gamma g) = \varphi_{\mathcal{M},q}(\gamma)\, \phi(g)$ for all $\gamma \in \Gamma_L$ and $g \in G$.
(L2) $\|\phi\|^2_{\mathcal{M},q} := \int_{\Gamma_L \backslash G} |\phi(\bar{g})|^2 \, d\bar{g} < \infty, \quad \bar{g} = \Gamma_L g.$

The induced representation $\pi_{\mathcal{M},q}$ is realized in $\mathcal{H}_{\mathcal{M},q}$ as follows:

$$\left(\pi_{\mathcal{M},q}(g_0)\phi\right)(g) := \phi(gg_0), \quad g_0, g \in G, \ \phi \in \mathcal{H}_{\mathcal{M},q}. \tag{3.11}$$

The representation $\pi_{\mathcal{M},q}$ is called the *lattice representation* of G associated with the lattice L.

Main Theorem. Let $\mathcal{M}$ be a positive definite, symmetric half integral matrix of degree h. Let $\varphi_{\mathcal{M}}$ be the character of Γ_L defined by $\varphi_{\mathcal{M}}((\lambda,\mu,\kappa)) := e^{2\pi i\sigma(\mathcal{M}\kappa)}$ for all $(\lambda,\mu,\kappa) \in \Gamma_L$. Then the lattice representation

$$\pi_{\mathcal{M}} := \operatorname{Ind}_{\Gamma_L}^G \varphi_{\mathcal{M}}$$

induced from the character $\varphi_{\mathcal{M}}$ is unitarily equivalent to the direct sum

$$\bigoplus U(\sigma_{\mathcal{M}}) := \bigoplus \operatorname{Ind}_K^G \sigma_{\mathcal{M}} \quad (\,(\det 2\mathcal{M}\,)^g\text{-copies}\,)$$

of the Schrödinger representation $\operatorname{Ind}_K^G \sigma_{\mathcal{M}}$.

Proof. We first recall that the induced representation $\pi_{\mathcal{M}}$ is realized in the Hilbert space $\mathcal{H}_{\mathcal{M}}$ consisting of all measurable functions $\phi : G \longrightarrow \mathbb{C}$ satisfying the conditions

$$\phi((\lambda_0,\mu_0,\kappa_0)\circ g) = e^{2\pi i\sigma(\mathcal{M}\kappa_0)}\,\phi(g), \quad (\lambda_0,\mu_0,\kappa_0) \in \Gamma_L, \ g \in G \tag{3.13}$$

and

$$\|\phi\|^2_{\pi,\mathcal{M}} := \int_{\Gamma_L\backslash G} |\phi(\bar g)|^2\, d\bar g < \infty, \quad \bar g = \Gamma_L \circ g. \tag{3.14}$$

Now, we write

$$g_0 = [\lambda_0,\mu_0,\kappa_0] \in \Gamma_L \ \ and \ \ g = [\lambda,\mu,\kappa] \in G.$$

For $\phi \in \mathcal{H}_{\mathcal{M}}$, we have

$$\phi(g_0 \diamond g) = \phi([\lambda_0+\lambda, \mu_0+\mu, \kappa_0+\kappa+\lambda_0\,{}^t\mu + \mu\,{}^t\lambda_0]). \tag{3.15}$$

On the other hand, we get

$$\begin{aligned}\phi(g_0 \diamond g) &= \phi((\lambda_0,\mu_0,\kappa_0-\mu_0\,{}^t\lambda_0)\circ g)\\ &= e^{2\pi i\sigma\{\mathcal{M}(\kappa_0-\mu_0\,{}^t\lambda_0)\}}\,\phi(g)\\ &= e^{2\pi i\sigma(\mathcal{M}\kappa_0)}\,\phi(g) \quad (\,\text{because } \sigma(\mathcal{M}\mu_0\,{}^t\lambda_0) \in \mathbb{Z}\,)\end{aligned}$$

Thus, putting $\kappa' := \kappa_0 + \lambda_0\,{}^t\mu + \mu\,{}^t\lambda_0$, we get

$$\phi([\lambda_0+\lambda, \mu_0+\mu, \kappa+\kappa']) = e^{2\pi i\sigma(\mathcal{M}\kappa')}\cdot e^{-4\pi i\sigma(\mathcal{M}\lambda_0\,{}^t\mu)}\,\phi([\lambda,\mu,\kappa]). \tag{3.16}$$

Putting $\lambda_0 = \kappa_0 = 0$ in (3.16), we have

$$(3.17)\quad \phi([\lambda, \mu+\mu_0, \kappa]) = \phi([\lambda, \mu, \kappa]) \text{ for all } \mu_0 \in \mathbb{Z}^{(h,g)} \text{ and } [\lambda, \mu, \kappa] \in G.$$

Therefore if we fix λ and κ, ϕ is periodic in μ with respect to the lattice $\mathbb{Z}^{(h,g)}$ in $\mathbb{R}^{(h,g)}$. We note that

$$\phi([\lambda, \mu, \kappa]) = \phi([0, 0, \kappa] \diamond [\lambda, \mu, 0]) = e^{2\pi i \sigma(\mathcal{M}\kappa)}\, \phi([\lambda, \mu, 0])$$

for $[\lambda, \mu, \kappa] \in G$. Hence, ϕ admits a Fourier expansion in μ :

$$(3.18)\qquad \phi([\lambda, \mu, \kappa]) = e^{2\pi i \sigma(\mathcal{M}\kappa)} \sum_{N \in \mathbb{Z}^{(h,g)}} c_N(\lambda)\, e^{2\pi i \sigma(N\, {}^t\mu)}.$$

If $\lambda_0 \in \mathbb{Z}^{(h,g)}$, then we have

$$\begin{aligned}
\phi([\lambda+\lambda_0, \mu, \kappa]) &= e^{2\pi i \sigma(\mathcal{M}\kappa)} \sum_{N \in \mathbb{Z}^{(h,g)}} c_N(\lambda+\lambda_0)\, e^{2\pi i \sigma(N\, {}^t\mu)} \\
&= e^{-4\pi i \sigma(\mathcal{M}\lambda_0\, {}^t\mu)}\, \phi([\lambda, \mu, \kappa]) \qquad (\text{by } (3.16)) \\
&= e^{-4\pi i \sigma(\mathcal{M}\lambda_0\, {}^t\mu)}\, e^{2\pi i \sigma(\mathcal{M}\kappa)} \sum_{N \in \mathbb{Z}^{(h,g)}} c_N(\lambda)\, e^{2\pi i \sigma(N\, {}^t\mu)}, \\
&= e^{2\pi i \sigma(\mathcal{M}\kappa)} \sum_{N \in \mathbb{Z}^{(h,g)}} c_N(\lambda)\, e^{2\pi i \sigma\{(N-2\mathcal{M}\lambda_0)\, {}^t\mu\}}. \quad (\text{by } (3.18))
\end{aligned}$$

So we get

$$\begin{aligned}
&\sum_{N \in \mathbb{Z}^{(h,g)}} c_N(\lambda+\lambda_0)\, e^{2\pi i \sigma(N\, {}^t\mu)} \\
&= \sum_{N \in \mathbb{Z}^{(h,g)}} c_N(\lambda)\, e^{2\pi i \sigma\{(N-2\mathcal{M}\lambda_0)\, {}^t\mu\}} \\
&= \sum_{N \in \mathbb{Z}^{(h,g)}} c_{N+2\mathcal{M}\lambda_0}(\lambda)\, e^{2\pi i \sigma(N\, {}^t\mu)}.
\end{aligned}$$

Hence, we get

$$(3.19)\qquad c_N(\lambda+\lambda_0) = c_{N+2\mathcal{M}\lambda_0}(\lambda) \text{ for all } \lambda_0 \in \mathbb{Z}^{(h,g)} \text{ and } \lambda \in \mathbb{R}^{(h,g)}.$$

Consequently, it is enough to know only the coefficients $c_\alpha(\lambda)$ for the representatives α in $\mathbb{Z}^{(h,g)}$ modulo $2\mathcal{M}$. It is obvious that the number of all such α's is $(\det 2\mathcal{M})^g$. We denote by $\mathcal{J}$ a complete system of such representatives in $\mathbb{Z}^{(h,g)}$ modulo $2\mathcal{M}$.

Then, we have

$$\phi([\lambda,\mu,\kappa]) = e^{2\pi i\sigma(\mathcal{M}\kappa)}\left\{\sum_{N\in\mathbb{Z}^{(h,g)}} c_{\alpha+2\mathcal{M}N}(\lambda)\, e^{2\pi i\sigma\{(\alpha+2\mathcal{M}N)\,{}^t\mu\}} + \sum_{N\in\mathbb{Z}^{(h,g)}} c_{\beta+2\mathcal{M}N}(\lambda)\, e^{2\pi i\sigma\{(\beta+2\mathcal{M}N)\,{}^t\mu\}} \right.$$
$$\cdot$$
$$\cdot$$
$$\left. + \sum_{N\in\mathbb{Z}^{(h,g)}} c_{\gamma+2\mathcal{M}N}(\lambda)\, e^{2\pi i\{(\gamma+2\mathcal{M}N)\,{}^t\mu\}}\right\},$$

where $\{\alpha,\beta,\cdots,\gamma\}$ denotes the complete system $\mathcal{J}$.

For each $\alpha\in\mathcal{J}$, we denote by $\mathcal{H}_{\mathcal{M},\alpha}$ the Hilbert space consisting of Fourier expansions

$$e^{2\pi i\sigma(\mathcal{M}\kappa)}\sum_{N\in\mathbb{Z}^{(h,g)}} c_{\alpha+2\mathcal{M}N}(\lambda)\, e^{2\pi i\sigma\{(\alpha+2\mathcal{M}N)\,{}^t\mu\}},\quad (\lambda,\mu,\kappa)\in G,$$

where $c_N(\lambda)$ denotes the coefficients of the Fourier expansion (3.18) of $\phi\in\mathcal{H}_{\mathcal{M}}$ and ϕ runs over the set $\{\phi\in\pi_{\mathcal{M}}\}$. It is easy to see that $\mathcal{H}_{\mathcal{M},\alpha}$ is invariant under $\pi_{\mathcal{M}}$. We denote the restriction of $\pi_{\mathcal{M}}$ to $\mathcal{H}_{\mathcal{M},\alpha}$ by $\pi_{\mathcal{M},\alpha}$. Then we have

$$\pi_{\mathcal{M}} = \bigoplus_{\alpha\in\mathcal{J}} \pi_{\mathcal{M},\alpha}. \tag{3.20}$$

Let $\phi_\alpha\in\pi_{\mathcal{M},\alpha}$. Then for $[\lambda,\mu,\kappa]\in G$, we get

$$\phi_\alpha([\lambda,\mu,\kappa]) = e^{2\pi i\sigma(\mathcal{M}\kappa)}\sum_{N\in\mathbb{Z}^{(h,g)}} c_{\alpha+2\mathcal{M}N}(\lambda)\, e^{2\pi i\sigma\{(\alpha+2\mathcal{M}N)\,{}^t\mu\}}. \tag{3.21}$$

We put

$$I_\lambda := \overbrace{[0,1]\times[0,1]\times\cdots\times[0,1]}^{(h\times g)\text{-times}} \subset \left\{[\lambda,0,0]\mid \lambda\in\mathbb{R}^{(h,g)}\right\}$$

and

$$I_\mu := \overbrace{[0,1]\times[0,1]\times\cdots\times[0,1]}^{(h\times g)\text{-times}} \subset \left\{[0,\mu,0]\mid \mu\in\mathbb{R}^{(h,g)}\right\}.$$

Then, we obtain

$$\int_{I_\mu}\phi_\alpha([\lambda,\mu,\kappa])\, e^{-2\pi i\sigma(\alpha\,{}^t\mu)}\,d\mu = e^{2\pi i\sigma(\mathcal{M}\kappa)}\,c_\alpha(\lambda),\quad \alpha\in\mathcal{J}. \tag{3.22}$$

Since $\Gamma_L\backslash G \cong I_\lambda \times I_\mu$, we get

$$\begin{aligned}\|\phi_\alpha\|^2_{\pi,\mathcal{M},\alpha} :&= \|\phi_\alpha\|^2_{\pi,\mathcal{M}} = \int_{\Gamma_L\backslash G} |\phi_\alpha(\bar{g})|^2\, d\bar{g} \\ &= \int_{I_\lambda}\int_{I_\mu} |\phi_\alpha(\bar{g})|^2\, d\lambda d\mu \\ &= \int_{I_\lambda\times I_\mu} \Big| \sum_{N\in\mathbb{Z}^{(h,g)}} c_{\alpha+2\mathcal{M}N}(\lambda)\, e^{2\pi i\sigma\{(\alpha+2\mathcal{M}N)\,{}^t\mu\}} \Big|^2 d\lambda d\mu \\ &= \int_{I_\lambda} \sum_{N\in\mathbb{Z}^{(h,g)}} |c_{\alpha+2\mathcal{M}N}(\lambda)|^2\, d\lambda \\ &= \int_{I_\lambda} \sum_{N\in\mathbb{Z}^{(h,g)}} |c_\alpha(\lambda+N)|^2\, d\lambda \qquad (\text{by } (3.19)) \\ &= \int_{\mathbb{R}^{(h,g)}} |c_\alpha(\lambda)|^2\, d\lambda.\end{aligned}$$

Since $\phi_\alpha \in \pi_{\mathcal{M},\alpha}$, $\|\phi_\alpha\|_{\pi,\mathcal{M},\alpha} < \infty$ and so $c_\alpha(\lambda) \in L^2\left(\mathbb{R}^{(h,g)}, d\xi\right)$ for all $\alpha \in \mathcal{J}$.

For each $\alpha \in \mathcal{J}$, we define the mapping $\vartheta_{\mathcal{M},\alpha}$ on $L^2\left(\mathbb{R}^{(h,g)}, d\xi\right)$ by

$$(3.23)\quad (\vartheta_{\mathcal{M},\alpha} f)([\lambda,\mu,\kappa]) := e^{2\pi i\sigma(\mathcal{M}\kappa)} \sum_{N\in\mathbb{Z}^{(h,g)}} f(\lambda+N)\, e^{2\pi i\sigma\{(\alpha+2\mathcal{M}N)\,{}^t\mu\}},$$

where $f \in L^2\left(\mathbb{R}^{(h,g)}, d\xi\right)$ and $[\lambda,\mu,\kappa] \in G$.

Lemma 3.4. For each $\alpha \in \mathcal{J}$, the image of $L^2\left(\mathbb{R}^{(h,g)}, d\xi\right)$ under $\vartheta_{\mathcal{M},\alpha}$ is contained in $\mathcal{H}_{\mathcal{M},\alpha}$. Moreover, the mapping $\vartheta_{\mathcal{M},\alpha}$ is a one-to-one unitary operator of $L^2\left(\mathbb{R}^{(h,g)}, d\xi\right)$ onto $\mathcal{H}_{\mathcal{M},\alpha}$ preserving the norms. In other words, the mapping

$$\vartheta_{\mathcal{M},\alpha} : L^2\left(\mathbb{R}^{(h,g)}, d\xi\right) \longrightarrow \mathcal{H}_{\mathcal{M},\alpha}$$

is an isometry.

Proof. We already showed that $\vartheta_{\mathcal{M},\alpha}$ preserves the norms. First, we observe that if $(\lambda_0,\mu_0,\kappa_0) \in \Gamma_L$ and $g = [\lambda,\mu,\kappa] \in G$,

$$\begin{aligned}(\lambda_0,\mu_0,\kappa_0)\circ g &= [\lambda_0,\mu_0,\kappa_0+\mu_0\,{}^t\lambda_0] \diamond [\lambda,\mu,\kappa] \\ &= [\lambda_0+\lambda,\mu_0+\mu,\kappa+\kappa_0+\mu_0\,{}^t\lambda_0+\lambda_0\,{}^t\mu+\mu\,{}^t\lambda_0].\end{aligned}$$

Thus we get

$$\begin{aligned}
&(\vartheta_{\mathcal{M},\alpha} f)((\lambda_0,\mu_0,\kappa_0)\circ g)\\
&= e^{2\pi i\sigma\{\mathcal{M}(\kappa+\kappa_0+\mu_0\,{}^t\lambda_0+\lambda_0\,{}^t\mu+\mu\,{}^t\lambda_0)\}} \sum_{N\in\mathbb{Z}^{(h,g)}} f(\lambda+\lambda_0+N)\, e^{2\pi i\{(\alpha+2\mathcal{M}N)\,{}^t(\mu_0+\mu)\}}\\
&= e^{2\pi i\sigma(\mathcal{M}\kappa_0)}\cdot e^{2\pi i\sigma(\mathcal{M}\kappa)}\cdot e^{2\pi i\sigma(\alpha\,{}^t\mu_0)} \sum_{N\in\mathbb{Z}^{(h,g)}} f(\lambda+N)\, e^{2\pi i\sigma\{(\alpha+2\mathcal{M}N)\,{}^t\mu\}}\\
&= e^{2\pi i\sigma(\mathcal{M}\kappa_0)}\,(\vartheta_{\mathcal{M},\alpha} f)(g).
\end{aligned}$$

Here, in the above equalities we used the facts that $2\sigma(\mathcal{M}N\,{}^t\mu_0)\in\mathbb{Z}$ and $\alpha\,{}^t\mu_0\in\mathbb{Z}$. It is easy to show that

$$\int_{\Gamma_L\backslash G} |\vartheta_{\mathcal{M},\alpha} f(\bar g)|^2\, d\bar g = \int_{\mathbb{R}^{(h,g)}} |f(\lambda)|^2\, d\lambda = \|f\|_2^2 < \infty.$$

This completes the proof of Lemma 3.4.

Finally, it is easy to show that for each $\alpha\in\mathcal{J}$, the mapping $\vartheta_{\mathcal{M},\alpha}$ intertwines the Schrödinger representation $\big(U(\sigma_{\mathcal{M}}), L^2(\mathbb{R}^{(h,g)}, d\xi)\big)$ and the representation $(\pi_{\mathcal{M},\alpha}, \mathcal{H}_{\mathcal{M},\alpha})$. Therefore, by Lemma 3.4, for each $\alpha\in\mathcal{J}$, $\pi_{\mathcal{M},\alpha}$ is unitarily equivalent to $U(\sigma_{\mathcal{M}})$ and so $\pi_{\mathcal{M},\alpha}$ is an irreducible unitary representation of G. According to (3.20), the induced representation $\pi_{\mathcal{M}}$ is unitarily equivalent to

$$\bigoplus U(\sigma_{\mathcal{M}}) \quad (\,(\det 2\mathcal{M})^g\text{-copies}\,).$$

This completes the proof of the Main Theorem. □

4. Relation of lattice representations to theta functions

In this section, we state the connection between lattice representations and theta functions. As before, we write $V=\mathbb{R}^{(h,g)}\times\mathbb{R}^{(h,g)}\cong\mathbb{C}^{(h,g)},\ L=\mathbb{Z}^{(h,g)}\times\mathbb{Z}^{(h,g)}$ and $\mathcal{M}$ is a positive symmetric half-integral matrix of degree h. The function $q_{\mathcal{M}}: L\longrightarrow\mathbb{R}/2\mathbb{Z}=[0,2)$ defined by

$$q_{\mathcal{M}}((\xi,\eta)) := 2\,\sigma(\mathcal{M}\xi\,{}^t\eta),\quad (\xi,\eta)\in L \tag{4.1}$$

satisfies Condition (3.8). We let $\varphi_{\mathcal{M},q_{\mathcal{M}}}:\Gamma_L\longrightarrow\mathbb{C}_1^{\times}$ be the character of Γ_L defined by

$$\varphi_{\mathcal{M},q_{\mathcal{M}}}((l,\kappa)) = e^{2\pi i\,\sigma(\mathcal{M}\kappa)}\, e^{\pi i\, q_{\mathcal{M}}(l)},\quad (l,\kappa)\in\Gamma_L.$$

We denote by $\mathcal{H}_{\mathcal{M},q_{\mathcal{M}}}$ the Hilbert space consisting of measurable functions $\phi: G\longrightarrow\mathbb{C}$ which satisfy Condition (4.2) and Condition (4.3):

$$\phi((l,\kappa)\circ g) = \varphi_{\mathcal{M},q_{\mathcal{M}}}((l,\kappa))\,\phi(g)\quad \text{for all } (l,\kappa)\in\Gamma_L \text{ and } g\in G. \tag{4.2}$$

(4.3) $$\int_{\Gamma_L\backslash G} \|\phi(\dot g)\|^2\, d\dot g \,<\, \infty, \quad \dot g = \Gamma_L \circ g.$$

Then the lattice representation

$$\pi_{\mathcal{M},q_{\mathcal{M}}} := \mathrm{Ind}_{\Gamma_L}^{G}\, \varphi_{\mathcal{M},q_{\mathcal{M}}}$$

of G induced from the character $\varphi_{\mathcal{M},q_{\mathcal{M}}}$ is realized in $\mathcal{H}_{\mathcal{M},q_{\mathcal{M}}}$ as

$$\big(\pi_{\mathcal{M},q_{\mathcal{M}}}(g_0)\,\phi\big)\,(g) \,=\, \phi(gg_0), \quad g_0, g \in G,\ \phi \in \mathcal{H}_{\mathcal{M},q_{\mathcal{M}}}.$$

Let $\mathbf{H}_{\mathcal{M},q_{\mathcal{M}}}$ be the vector space consisting of measurable functions $F : V \longrightarrow \mathbb{C}$ satisfying Conditions (4.4) and (4.5).

(4.4) $$F(\lambda+\xi, \mu+\eta) \,=\, e^{2\pi i\, \sigma\{\mathcal{M}(\xi\, {}^t\eta+\lambda\, {}^t\eta-\mu\, {}^t\xi)\}}\, F(\lambda,\mu)$$

for all $(\lambda,\mu) \in V$ and $(\xi,\eta) \in L$.

(4.5) $$\int_{L\backslash V} \|F(\dot v)\|^2\, d\dot v \,=\, \int_{I_\lambda\times I_\mu} \|F(\lambda,\mu)\|^2\, d\lambda d\mu \,<\, \infty.$$

Given $\phi \in \mathcal{H}_{\mathcal{M},q_{\mathcal{M}}}$ and a fixed element $\Omega \in H_g$, we put

(4.6) $$E_\phi(\lambda,\mu) := \phi((\lambda,\mu,0)), \quad \lambda,\mu \in \mathbb{R}^{(h,g)},$$

(4.7) $$F_\phi(\lambda,\mu) := \phi([\lambda,\mu,0]), \quad \lambda,\mu \in \mathbb{R}^{(h,g)},$$

(4.8) $$F_{\Omega,\phi}(\lambda,\mu) := e^{-2\pi i\, \sigma(\mathcal{M}\lambda\Omega\, {}^t\lambda)}\, F_\phi(\lambda,\mu), \quad \lambda,\mu \in \mathbb{R}^{(h,g)}.$$

In addition, we put for $W = \lambda\Omega+\mu \in \mathbb{C}^{(h,g)}$,

(4.9) $$\vartheta_{\Omega,\phi}(W) = \vartheta_{\Omega,\phi}(\lambda\Omega+\mu) := F_{\Omega,\phi}(\lambda,\mu).$$

We observe that E_ϕ, F_ϕ and $F_{\Omega,\phi}$ are functions defined on V and $\vartheta_{\Omega,\phi}$ is a function defined on $\mathbb{C}^{(h,g)}$.

Proposition 4.1. If $\phi \in \mathcal{H}_{\mathcal{M},q_{\mathcal{M}}}$, $(\xi,\eta) \in L$ and $(\lambda,\mu) \in V$, then we have the formulas

(4.10) $$E_\phi(\lambda+\xi,\mu+\eta) \,=\, e^{2\pi i\, \sigma\{\mathcal{M}(\xi\, {}^t\eta+\lambda\, {}^t\eta-\mu\, {}^t\xi)\}}\, E_\phi(\lambda,\mu).$$

(4.11) $$F_\phi(\lambda+\xi,\mu+\eta) \,=\, e^{-4\pi i\, \sigma(\mathcal{M}\xi\, {}^t\mu)}\, F_\phi(\lambda,\mu).$$

(4.12) $$F_{\Omega,\phi}(\lambda+\xi,\mu+\eta) \,=\, e^{-2\pi i\, \sigma\{\mathcal{M}(\xi\Omega\, {}^t\xi+2\lambda\Omega\, {}^t\xi+2\mu\, {}^t\xi)\}}\, F_{\Omega,\phi}(\lambda,\mu).$$

If $W = \lambda\Omega+\eta \in \mathbb{C}^{(h,g)}$, then we have

(4.13) $$\vartheta_{\Omega,\phi}(W+\xi\Omega+\eta) \,=\, e^{-2\pi i\, \sigma\{\mathcal{M}(\xi\Omega\, {}^t\xi+2W\, {}^t\xi)\}}\, \vartheta_{\Omega,\phi}(W).$$

Moreover, F_ϕ is an element of $\mathbf{H}_{\mathcal{M},q_{\mathcal{M}}}$.

Proof. We note that

$$(\lambda+\xi,\mu+\eta,0)=(\xi,\eta,-\xi\,{}^t\mu+\eta\,{}^t\lambda)\circ(\lambda,\mu,0).$$

Thus we have

$$\begin{aligned}E_\phi(\lambda+\xi,\mu+\eta)&=\phi((\lambda+\xi,\mu+\eta,0))\\&=\phi((\xi,\eta,\,-\xi\,{}^t\mu+\eta\,{}^t\lambda)\circ(\lambda,\mu,0))\\&=\,e^{2\pi i\sigma\{\mathcal{M}(\xi\,{}^t\eta+\lambda\,{}^t\eta-\mu\,{}^t\xi)\}}\,\phi((\lambda,\mu,0))\\&=\,e^{2\pi i\sigma\{\mathcal{M}(\xi\,{}^t\eta+\lambda\,{}^t\eta-\mu\,{}^t\xi)\}}\,E_\phi(\lambda,\mu).\end{aligned}$$

This proves Formula (4.10). We observe that

$$[\lambda+\xi,\mu+\eta,0]=(\xi,\eta,\,-\xi\,{}^t\mu-\mu\,{}^t\xi-\eta\,{}^t\xi)\circ[\lambda,\mu,0].$$

Thus we have

$$\begin{aligned}F_\phi(\lambda+\xi,\mu+\eta)&=\phi([\lambda+\xi,\mu+\eta,0])\\&=\,e^{-2\pi i\,\sigma\{\mathcal{M}(\xi\,{}^t\mu+\mu\,{}^t\xi+\eta\,{}^t\xi)\}}\\&\qquad\times e^{2\pi i\,\sigma(\mathcal{M}\xi\,{}^t\eta)}\,\phi([\lambda,\mu,0])\\&=\,e^{-4\pi i\,\sigma(\mathcal{M}\xi\,{}^t\mu)}\,\phi([\lambda,\mu,0])\\&=\,e^{-4\pi i\,\sigma(\mathcal{M}\xi\,{}^t\mu)}\,F_\phi(\lambda,\mu).\end{aligned}$$

This proves Formula (4.11). According to (4.11), we have

$$\begin{aligned}F_{\Omega,\phi}(\lambda+\xi,\mu+\eta)&=\,e^{-2\pi i\,\sigma\{\mathcal{M}(\lambda+\xi)\Omega\,{}^t(\lambda+\xi)\}}\,F_\phi(\lambda+\xi,\mu+\eta)\\&=\,e^{-2\pi i\,\sigma\{\mathcal{M}(\lambda+\xi)\Omega\,{}^t(\lambda+\xi)\}}\\&\qquad\times e^{-4\pi i\,\sigma(\mathcal{M}\xi\,{}^t\mu)}\,F_\phi(\lambda,\mu)\\&=\,e^{-2\pi i\,\sigma\{\mathcal{M}(\xi\Omega\,{}^t\xi+2\lambda\Omega\,{}^t\xi+2\mu\,{}^t\xi)\}}\\&\qquad\times e^{-2\pi i\,\sigma(\mathcal{M}\lambda\Omega\,{}^t\lambda)}\,F_\phi(\lambda,\mu)\\&=\,e^{-2\pi i\,\sigma\{\mathcal{M}(\xi\Omega\,{}^t\xi+2\lambda\Omega\,{}^t\xi+2\mu\,{}^t\xi)\}}\,F_{\Omega,\phi}(\lambda,\mu).\end{aligned}$$

This proves Formula (4.12). Formula (4.13) follows immediately from Formula (4.12). Indeed, if $W=\lambda\Omega+\mu$ with $\lambda,\mu\in\mathbb{R}^{(h,g)}$, we have

$$\begin{aligned}\vartheta_{\Omega,\phi}(W+\xi\Omega+\eta)&=F_{\Omega,\phi}(\lambda+\xi,\mu+\eta)\\&=\,e^{-2\pi i\,\sigma\{\mathcal{M}(\xi\Omega\,{}^t\xi+2(\lambda\Omega+\mu)\,{}^t\xi)\}}\,F_{\Omega,\phi}(\lambda,\mu)\\&=\,e^{-2\pi i\,\sigma\{\mathcal{M}(\xi\Omega\,{}^t\xi+2W\,{}^t\xi)\}}\,\vartheta_{\Omega,\phi}(W).\end{aligned}$$

□

Remark 4.2. The function $\vartheta_{\Omega,\phi}(W)$ is a theta function of level $2\mathcal{M}$ with respect to Ω if $\vartheta_{\Omega,\phi}$ is holomorphic. For any $\phi \in \mathcal{H}_{\mathcal{M},q_{\mathcal{M}}}$, the function $\vartheta_{\Omega,\phi}$ satisfies the well known transformation law of a theta function. In this sense, the lattice representation $(\pi_{\mathcal{M},q_{\mathcal{M}}}, \mathcal{H}_{\mathcal{M},q_{\mathcal{M}}})$ is closely related to theta functions.

References

[B] R. Berndt, Darstellungen der Heisenberggruppe und Thetafunktionen. Vorlesungsausarbeitung, Hamburg, 1988

[C] P. Cartier, Quantum Mechanical Commutation Relations and Theta Functions, Proc. of Symp. Pure Mathematics, **9** Amer. Math. Soc., 1966, pp. 361–383

[F-C] G. Faltings and C.-L. Chai, Degeneration of Abelian Varieties, EMG, Band **22**, Springer-Verlag, New York/Berlin, 1990

[I] J. Igusa, Theta functions, Springer-Verlag, New York/Berlin, 1972

[M] G.W. Mackey, Induced Representations of Locally Compact Groups I, Ann. of Math. **55** (1952), 101–139

[N] Y. Namikawa, Toroidal Compactification of Siegel Spaces, Lect. Notes in Math. 812, Springer-Verlag, New York/Berlin, 1980

[Y1] J.-H. Yang, Harmonic Analysis on the Quotient Spaces of Heisenberg Groups, Nagoya Math. J. **123** (1991), 103–117

[Y2] J.-H. Yang, Harmonic Analysis on the Quotient Spaces of Heisenberg Groups, II, J. Number Theory **49** (1994), 63–72

[Y3] J.-H. Yang, A decomposition theorem on differential polynomials of theta functions of high level, Japanese J. Math., Math. Soc. Japan, New Series **22** (1996), 37–49

[Y4] J.-H. Yang, The Siegel-Jacobi Operator, Abh. Math. Sem. Univ. Hamburg **63** (1993), 135–146

[Y5] J.-H. Yang, Remarks on Jacobi forms of higher degree, Proc. of the 1993 Workshop on Automorphic Forms and Related Topics, PIMS (1993), 33–58

[Y6] J.-H. Yang, Singular Jacobi Forms, Trans. Amer. Math. Soc. **347** (1995), 2041–2049

[Y7] J.-H. Yang, Construction of Vector-Valued Modular Forms from Jacobi Forms, Canadian J. Math. **47** (1995), 1329–1339

[Y8] J.-H. Yang, A geometrical theory of Jacobi forms of higher degree, Proc. of Symposium on Hodge Theory and Algebraic Geometry (edited by Tadao Oda), Sendai, Japan (1996), 125–147

[Z] C. Ziegler, Jacobi Forms of Higher Degree, Abh. Math. Sem. Univ. Hamburg **59** (1989), 191–224