THE SCHRÖDINGER-WEIL REPRESENTATION AND JACOBI FORMS OF HALF INTEGRAL WEIGHT

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Abstract. In this paper, we define the concept of Jacobi forms of half integral weight using Takase's automorphic factor of weight 1/2 for a two-fold covering group of the symplectic group $Sp(n,\mathbb{R})$ on the Siegel upper half plane and find covariant maps for the Schrödinger-Weil representation. Using these covariant maps, we construct Jacobi forms of half integral weight with respect to an arithmetic subgroup of the Jacobi group.

1. Introduction

For a given fixed positive integer n, we let

$$\mathbb{H}_n = \left\{ \Omega \in \mathbb{C}^{(n,n)} \,\middle|\, \Omega = {}^t\Omega, \quad \mathrm{Im}\,\Omega > 0 \right\}$$

be the Siegel upper half plane of degree n and let

$$Sp(n,\mathbb{R}) = \left\{ g \in \mathbb{R}^{(2n,2n)} \,\middle|\, {}^t g J_n g = J_n \right\}$$

be the symplectic group of degree n, where $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F for two positive integers k and l, ${}^tM$ denotes the transpose of a matrix M, $\mathrm{Im}\,\Omega$ denotes the imaginary part of Ω and

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Here I_n denotes the identity matrix of degree n. We see that $Sp(n,\mathbb{R})$ acts on $\mathbb{H}_n$ transitively by

$$g \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1},$$

where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R})$ and $\Omega \in \mathbb{H}_n$.

For two positive integers n and m, we consider the Heisenberg group

$$H_{\mathbb{R}}^{(n,m)} = \{ (\lambda,\mu;\kappa) \mid \lambda,\mu \in \mathbb{R}^{(m,n)},\ \kappa \in \mathbb{R}^{(m,m)},\ \ \kappa + \mu\,{}^t\lambda \text{ symmetric } \}$$

endowed with the following multiplication law

$$(\lambda,\mu;\kappa) \circ (\lambda',\mu';\kappa') = (\lambda+\lambda', \mu+\mu'; \kappa+\kappa' + \lambda\,{}^t\mu' - \mu\,{}^t\lambda').$$

We let

$$G^J = Sp(n,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)} \qquad \text{(semi-direct product)}$$

be the Jacobi group endowed with the following multiplication law

$$\Big(g,(\lambda,\mu;\kappa)\Big) \cdot \Big(g',(\lambda',\mu';\kappa')\Big) = \Big(gg', (\widetilde{\lambda}+\lambda', \widetilde{\mu}+\mu'; \kappa+\kappa' + \widetilde{\lambda}\,{}^t\mu' - \widetilde{\mu}\,{}^t\lambda')\Big)$$

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with $g, g' \in Sp(n,\mathbb{R}), (\lambda,\mu;\kappa), (\lambda',\mu';\kappa') \in H_{\mathbb{R}}^{(n,m)}$ and $(\widetilde{\lambda},\widetilde{\mu}) = (\lambda,\mu)g'$. We let $\Gamma_n = Sp(n,\mathbb{Z})$ be the Siegel modular group of degree n. We let

$$\Gamma_n^J = \Gamma_n \ltimes H_{\mathbb{Z}}^{(n,m)}$$

be the Jacobi modular group. Then we have the *natural action* of G^J on the Siegel-Jacobi space $\mathbb{H}_{n,m} := \mathbb{H}_n \times \mathbb{C}^{(m,n)}$ defined by

$$\Big(g,(\lambda,\mu;\kappa)\Big)\cdot(\Omega,Z) = \Big(g\cdot\Omega,(Z+\lambda\,\Omega+\mu)(C\,\Omega+D)^{-1}\Big), \tag{1.1}$$

where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R}),\ (\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)}$ and $(\Omega, Z) \in \mathbb{H}_{n,m}$. We refer to [2, 3], [4], [29]-[38] for more details on materials, e.g., Jacobi forms, invariant metrics, invariant differential operators, Maass-Jacobi forms etc related to the Siegel-Jacobi space.

The Weil representation for a symplectic group was first introduced by A. Weil in [23] to reformulate Siegel's analytic theory of quadratic forms (cf. [18]) in terms of the group theoretical theory. It is well known that the Weil representation plays a central role in the study of the transformation behaviors of theta series. Whenever we study the transformation formulas of theta series or Siegel modular forms of half integral weights, we are troubled by the ambiguity of the factor $\det(C\Omega+D)^{1/2}$ in its signature. This means that we should consider the transformation formula on a non-trivial two-fold covering group of a symplectic group. In his paper [21], Takase removed the ambiguity of the factor $\det(C\Omega+D)^{1/2}$ by constructing the *right explicit* automorphic factor $J_{1/2}$ of weight $1/2$ for $Sp(n,\mathbb{R})_*$ on $\mathbb{H}_n$:

$$J_{1/2} : Sp(n,\mathbb{R})_* \times \mathbb{H}_n \longrightarrow \mathbb{C}^*. \tag{1.2}$$

Here $Sp(n,\mathbb{R})_*$ is the two-fold covering group of $Sp(n,\mathbb{R})$ in the sense of a real Lie group. See (4.15) for the precise definition. $J_{1/2}$ is real analytic on $Sp(n,\mathbb{R})_*$, holomorphic on $\mathbb{H}_n$ and satisfies the relation

$$J_{1/2}(g_*,\Omega)^2 = \det(C\Omega+D), \tag{1.3}$$

where $g_* = (g,\epsilon) \in Sp(n,\mathbb{R})_*$ with $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R})$. Using the automorphic factor $J_{1/2}$ of weight $1/2$, Takase expressed the transformation formula of theta series without ambiguity of $\det(C\Omega+D)^{1/2}$. Moreover he decomposed the automorphic factor

$$j(\gamma,\Omega) = \frac{\vartheta(\gamma\cdot\Omega)}{\vartheta(\Omega)}$$

with a standard theta series $\vartheta(\Omega)$ into a product of a character and the automorphic factor $J_{1/2}(\gamma_*,\Omega)$. The automorphic factor $J_{1/2}$ of Takase will play an important role in the further study of half integral weight Siegel modular forms and half integral weight Jacobi forms.

This paper is organized as follows. In Section 2, we discuss the Schrödinger representation of the Heisenberg group $H_{\mathbb{R}}^{(n,m)}$ associated with a nonzero symmetric real matrix of degree m which is formulated in [24, 25]. In Section 3, we define the Schrödinger-Weil representation $\omega_{\mathcal{M}}$ of the Jacobi group G^J associated with a symmetric positive definite matrix $\mathcal{M}$ and provide some of the actions of $\omega_{\mathcal{M}}$ on the representation space $L^2\big(\mathbb{R}^{(m,n)}\big)$ explicitly. In Section 4, we review Jacobi forms of integral weight, Siegel modular forms of half integral weight, and define Jacobi forms of half integral weight using the automorphic factor $J_{1/2}$ of weight 1.2 for the metaplectic group $Sp(n,\mathbb{R})_*$ on the Siegel upper half plane. In Section 5, we find covariant maps for the Schrödinger-Weil representation $\omega_{\mathcal{M}}$. In the final section we construct Jacobi forms of half integral weight with respect to an arithmetic subgroup of Γ^J using covariant maps obtained in Section 5.

Notations : We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of integers, the field of real numbers and the field of complex numbers respectively. $\mathbb{C}^\times$ denotes the multiplicative group of nonzero complex numbers and $\mathbb{Z}^\times$ denotes the set of all nonzero integers.. T denotes the multiplicative group of complex numbers of modulus one. The symbol ":=" means that the expression on the right is the definition of that on the left. For two positive integers k and l, $F^{(k,l)}$ denotes the set of all $k\times l$ matrices with entries in a commutative ring F. For a square matrix $A\in F^{(k,k)}$ of degree k, $\sigma(A)$ denotes the trace of A. For any $M\in F^{(k,l)}$, ${}^tM$ denotes the transpose of a matrix M. I_n denotes the identity matrix of degree n. We put $i=\sqrt{-1}$. For $z\in\mathbb{C}$, we define $z^{1/2}=\sqrt{z}$ so that $-\pi/2<\ \arg(z^{1/2})\leqq\pi/2$. Further we put $z^{\kappa/2}=\big(z^{1/2}\big)^\kappa$ for every $\kappa\in\mathbb{Z}$. For a positive integer m we denote by $S(m,F)$ the additive group consisting of all $m\times m$ symmetric matrices with coefficients in a commutative ring F.

2. The Schrödinger Representation

First of all, we observe that $H_{\mathbb{R}}^{(n,m)}$ is a 2-step nilpotent Lie group. The inverse of an element $(\lambda,\mu;\kappa)\in H_{\mathbb{R}}^{(n,m)}$ is given by

$$(\lambda,\mu;\kappa)^{-1}=(-\lambda,-\mu;-\kappa+\lambda\,{}^t\mu-\mu\,{}^t\lambda).$$

Now we set

$$[\lambda,\mu;\kappa]=(0,\mu;\kappa)\circ(\lambda,0;0)=(\lambda,\mu;\kappa-\mu\,{}^t\lambda).$$

Then $H_{\mathbb{R}}^{(n,m)}$ may be regarded as a group equipped with the following multiplication

$$[\lambda,\mu;\kappa]\diamond[\lambda_0,\mu_0;\kappa_0]=[\lambda+\lambda_0,\mu+\mu_0;\kappa+\kappa_0+\lambda\,{}^t\mu_0+\mu_0\,{}^t\lambda].$$

The inverse of $[\lambda,\mu;\kappa]\in H_{\mathbb{R}}^{(n,m)}$ is given by

$$[\lambda,\mu;\kappa]^{-1}=[-\lambda,-\mu;-\kappa+\lambda\,{}^t\mu+\mu\,{}^t\lambda].$$

We set

$$L=\left\{\,[0,\mu;\kappa]\in H_{\mathbb{R}}^{(n,m)}\;\Big|\;\mu\in\mathbb{R}^{(m,n)},\ \kappa={}^t\kappa\in\mathbb{R}^{(m,m)}\,\right\}.$$

Then L is a commutative normal subgroup of $H_{\mathbb{R}}^{(n,m)}$. Let $\widehat{L}$ be the Pontrajagin dual of L, i.e., the commutative group consisting of all unitary characters of L. Then $\widehat{L}$ is isomorphic to the additive group $\mathbb{R}^{(m,n)}\times S(m,\mathbb{R})$ via

$$\langle a,\hat{a}\rangle=e^{2\pi i\sigma(\hat{\mu}\,{}^t\mu+\hat{\kappa}\kappa)},\quad a=[0,\mu;\kappa]\in L,\ \hat{a}=(\hat{\mu},\hat{\kappa})\in\widehat{L},$$

where $S(m,\mathbb{R})$ denotes the space of all symmetric $m\times m$ real matrices.

We put

$$S=\left\{\,[\lambda,0;0]\in H_{\mathbb{R}}^{(n,m)}\;\Big|\;\lambda\in\mathbb{R}^{(m,n)}\,\right\}\cong\mathbb{R}^{(m,n)}.$$

Then S acts on L as follows:

$$\alpha_\lambda([0,\mu;\kappa]) = [0,\mu;\kappa + \lambda\,{}^t\!\mu + \mu\,{}^t\!\lambda], \quad [\lambda,0,0]\in S.$$

We see that the Heisenberg group $\left(H_{\mathbb{R}}^{(n,m)},\diamond\right)$ is isomorphic to the semi-direct product $S\ltimes L$ of S and L whose multiplication is given by

$$(\lambda,a)\cdot(\lambda_0,a_0) = \big(\lambda+\lambda_0, a+\alpha_\lambda(a_0)\big), \quad \lambda,\lambda_0\in S,\ a,a_0\in L.$$

On the other hand, S acts on $\widehat{L}$ by

$$\alpha_\lambda^*(\widehat{a}) = (\hat{\mu}+2\hat{\kappa}\lambda, \hat{\kappa}), \quad [\lambda,0;0]\in S, \quad \widehat{a}=(\hat{\mu},\hat{\kappa})\in\widehat{L}.$$

Then, we have the relation $\langle \alpha_\lambda(a),\widehat{a}\rangle = \langle a, \alpha_\lambda^*(\widehat{a})\rangle$ for all $a\in L$ and $\widehat{a}\in\widehat{L}$.

We have three types of S-orbits in $\widehat{L}$.

Type I. Let $\hat{\kappa}\in S(m,\mathbb{R})$ be nondegenerate. The S-orbit of $\hat{a}(\hat{\kappa})=(0,\hat{\kappa})\in\widehat{L}$ is given by

$$\widehat{\mathcal{O}}_{\hat{\kappa}} = \left\{ (2\hat{\kappa}\lambda,\hat{\kappa})\in\widehat{L}\ \Big|\ \lambda\in\mathbb{R}^{(m,n)}\right\} \cong \mathbb{R}^{(m,n)}.$$

Type II. Let $(\hat{\mu},\hat{\kappa})\in\mathbb{R}^{(m,n)}\times S(m,\mathbb{R})$ with $\hat{\mu}\in\mathbb{R}^{(m,n)}$, $\hat{\kappa}\in S(m,\mathbb{R})$ and degenerate $\hat{\kappa}\neq 0$. Then

$$\widehat{\mathcal{O}}_{(\hat{\mu},\hat{\kappa})} = \left\{ (\hat{\mu}+2\hat{\kappa}\lambda,\hat{\kappa})\Big|\ \lambda\in\mathbb{R}^{(m,n)}\right\} \subsetneqq \mathbb{R}^{(m,n)}\times\{\hat{\kappa}\}.$$

Type III. Let $\hat{y}\in\mathbb{R}^{(m,n)}$. The S-orbit $\widehat{\mathcal{O}}_{\hat{y}}$ of $\hat{a}(\hat{y})=(\hat{y},0)$ is given by

$$\widehat{\mathcal{O}}_{\hat{y}} = \{\,(\hat{y},0)\,\} = \hat{a}(\hat{y}).$$

We have

$$\widehat{L} = \left(\bigcup_{\substack{\hat{\kappa}\in S(m,\mathbb{R})\\ \hat{\kappa}\ \text{nondegenerate}}} \widehat{\mathcal{O}}_{\hat{\kappa}} \right) \bigcup \left(\bigcup_{\hat{y}\in\mathbb{R}^{(m,n)}} \widehat{\mathcal{O}}_{\hat{y}} \right) \bigcup \left(\bigcup_{\substack{(\hat{\mu},\hat{\kappa})\in\mathbb{R}^{(m,n)}\times S(m,\mathbb{R})\\ \hat{\kappa}\neq 0\ \text{degenerate}}} \widehat{\mathcal{O}}_{(\hat{\mu},\hat{\kappa})} \right)$$

as a set. The stabilizer $S_{\hat{\kappa}}$ of S at $\hat{a}(\hat{\kappa})=(0,\hat{\kappa})$ is given by

$$S_{\hat{\kappa}} = \{0\}.$$

And the stabilizer $S_{\hat{y}}$ of S at $\hat{a}(\hat{y})=(\hat{y},0)$ is given by

$$S_{\hat{y}} = \left\{ [\lambda,0;0]\ \Big|\ \lambda\in\mathbb{R}^{(m,n)} \right\} = S \ \cong\ \mathbb{R}^{(m,n)}.$$

In this section, for the present being we set $H=H_{\mathbb{R}}^{(n,m)}$ for brevity. We see that L is a closed, commutative normal subgroup of H. Since $(\lambda,\mu;\kappa)=(0,\mu;\kappa+\mu\,{}^t\!\lambda)\circ(\lambda,0;0)$ for $(\lambda,\mu;\kappa)\in H$, the homogeneous space $X=L\backslash H$ can be identified with $\mathbb{R}^{(m,n)}$ via

$$Lh = L\circ(\lambda,0;0)\longmapsto\lambda, \quad h=(\lambda,\mu;\kappa)\in H.$$

We observe that H acts on X by

$$(Lh)\cdot h_0 = L\,(\lambda+\lambda_0,0;0) = \lambda+\lambda_0,$$

where $h=(\lambda,\mu;\kappa)\in H$ and $h_0=(\lambda_0,\mu_0;\kappa_0)\in H$.

If $h=(\lambda,\mu;\kappa)\in H$, we have

$$l_h = (0,\mu;\kappa+\mu\,{}^t\!\lambda), \quad s_h = (\lambda,0;0)$$

in the Mackey decomposition of $h = l_h \circ s_h$ (cf. [11]). Thus if $h_0 = (\lambda_0, \mu_0; \kappa_0) \in H$, then we have

$$s_h \circ h_0 = (\lambda, 0; 0) \circ (\lambda_0, \mu_0; \kappa_0) = (\lambda + \lambda_0, \mu_0; \kappa_0 + \lambda\, {}^t\mu_0)$$

and so

$$l_{s_h \circ h_0} = \big(0, \mu_0; \kappa_0 + \mu_0\, {}^t\lambda_0 + \lambda\, {}^t\mu_0 + \mu_0\, {}^t\lambda\big). \tag{2.1}$$

For a real symmetric matrix $c = {}^tc \in S(m, \mathbb{R})$ with $c \neq 0$, we consider the unitary character χ_c of L defined by

$$\chi_c\left((0, \mu; \kappa)\right) = e^{\pi i \sigma(c\kappa)}\, I, \quad (0, \mu; \kappa) \in L, \tag{2.2}$$

where I denotes the identity mapping. Then the representation $\mathscr{W}_c = \mathrm{Ind}_L^H\, \chi_c$ of H induced from χ_c is realized on the Hilbert space $H(\chi_c) = L^2\big(X, d\dot{h}, \mathbb{C}\big) \cong L^2\left(\mathbb{R}^{(m,n)}, d\xi\right)$ as follows. If $h_0 = (\lambda_0, \mu_0; \kappa_0) \in H$ and $x = Lh \in X$ with $h = (\lambda, \mu; \kappa) \in H$, we have

$$\left(\mathscr{W}_c(h_0)f\right)(x) = \chi_c(l_{s_h \circ h_0}) f(xh_0), \quad f \in H(\chi_c). \tag{2.3}$$

It follows from (2.1) that

$$\left(\mathscr{W}_c(h_0)f\right)(\lambda) = e^{\pi i \sigma\{c(\kappa_0 + \mu_0\, {}^t\lambda_0 + 2\lambda\, {}^t\mu_0)\}}\, f(\lambda + \lambda_0), \tag{2.4}$$

where $h_0 = (\lambda_0, \mu_0; \kappa_0) \in H$ and $\lambda \in \mathbb{R}^{(m,n)}$. Here we identified $x = Lh$ (resp. $xh_0 = Lhh_0$) with λ (resp. $\lambda + \lambda_0$). The induced representation $\mathscr{W}_c$ is called the Schrödinger representation of H associated with χ_c. Thus $\mathscr{W}_c$ is a monomial representation.

Theorem 2.1. *Let c be a positive definite symmetric real matrix of degree m. Then the Schrödinger representation $\mathscr{W}_c$ of H is irreducible.*

Proof. The proof can be found in [24], Theorem 3. □

Remark. We refer to [24]-[28] for more representations of the Heisenberg group $H_{\mathbb{R}}^{(n,m)}$ and their related topics.

3. The Schrödinger-Weil Representation

Throughout this section we assume that $\mathcal{M}$ is a symmetric real positive definite $m \times m$ matrix. We consider the Schrödinger representation $\mathscr{W}_{\mathcal{M}}$ of the Heisenberg group $H_{\mathbb{R}}^{(n,m)}$ with the central character $\mathscr{W}_{\mathcal{M}}((0,0;\kappa)) = \chi_{\mathcal{M}}((0,0;\kappa)) = e^{\pi i\, \sigma(\mathcal{M}\kappa)}$, $\kappa \in S(m, \mathbb{R})$ (cf. (2.2)). We note that the symplectic group $Sp(n, \mathbb{R})$ acts on $H_{\mathbb{R}}^{(n,m)}$ by conjugation inside G^J. For a fixed element $g \in Sp(n, \mathbb{R})$, the irreducible unitary representation $\mathscr{W}_{\mathcal{M}}^g$ of $H_{\mathbb{R}}^{(n,m)}$ defined by

$$\mathscr{W}_{\mathcal{M}}^g(h) = \mathscr{W}_{\mathcal{M}}(ghg^{-1}), \quad h \in H_{\mathbb{R}}^{(n,m)} \tag{3.1}$$

has the property that

$$\mathscr{W}_{\mathcal{M}}^g((0,0;\kappa)) = \mathscr{W}_{\mathcal{M}}((0,0;\kappa)) = e^{\pi i\, \sigma(\mathcal{M}\kappa)}\, \mathrm{Id}_{H(\chi_{\mathcal{M}})}, \quad \kappa \in S(m, \mathbb{R}).$$

Here $\mathrm{Id}_{H(\chi_{\mathcal{M}})}$ denotes the identity operator on the Hilbert space $H(\chi_{\mathcal{M}})$. According to Stone-von Neumann theorem, there exists a unitary operator $R_{\mathcal{M}}(g)$ on $H(\chi_{\mathcal{M}})$ such that

$$R_{\mathcal{M}}(g)\mathscr{W}_{\mathcal{M}}(h) = \mathscr{W}_{\mathcal{M}}^g(h) R_{\mathcal{M}}(g) \qquad \text{for all } h \in H_{\mathbb{R}}^{(n,m)}. \tag{3.2}$$

We observe that $R_{\mathcal{M}}(g)$ is determined uniquely up to a scalar of modulus one.

From now on, for brevity, we put $G = Sp(n,\mathbb{R})$. According to Schur's lemma, we have a map $c_{\mathcal{M}} : G \times G \longrightarrow T$ satisfying the relation

$$R_{\mathcal{M}}(g_1 g_2) = c_{\mathcal{M}}(g_1,g_2) R_{\mathcal{M}}(g_1) R_{\mathcal{M}}(g_2) \quad \text{for all} \quad g_1, g_2 \in G. \tag{3.3}$$

Therefore $R_{\mathcal{M}}$ is a projective representation of G on $H(\chi_{\mathcal{M}})$ and $c_{\mathcal{M}}$ defines the cocycle class in $H^2(G,T)$. The cocycle $c_{\mathcal{M}}$ yields the central extension $G_{\mathcal{M}}$ of G by T. The group $G_{\mathcal{M}}$ is a set $G \times T$ equipped with the following multiplication

$$(g_1,t_1)\cdot(g_2,t_2) = \big(g_1g_2, t_1t_2\, c_{\mathcal{M}}(g_1,g_2)^{-1/m}\big), \quad g_1,g_2 \in G,\ t_1,t_2 \in T. \tag{3.4}$$

We see immediately that the map $\widetilde{R}_{\mathcal{M}} : G_{\mathcal{M}} \longrightarrow GL(H(\chi_{\mathcal{M}}))$ defined by

$$\widetilde{R}_{\mathcal{M}}(g,t) = t^m\, R_{\mathcal{M}}(g) \quad \text{for all } (g,t) \in G_{\mathcal{M}} \tag{3.5}$$

is a true representation of $G_{\mathcal{M}}$. As in Section 1.7 in [10], we can define the map $s_{\mathcal{M}} : G \longrightarrow T$ satisfying the relation

$$c_{\mathcal{M}}(g_1,g_2)^2 = s_{\mathcal{M}}(g_1)^{-1} s_{\mathcal{M}}(g_2)^{-1} s_{\mathcal{M}}(g_1g_2) \quad \text{for all } g_1,g_2 \in G.$$

Thus we see that

$$G_{2,\mathcal{M}} = \Big\{ (g,t) \in G_{\mathcal{M}} \mid t^2 = s_{\mathcal{M}}(g)^{-1/m} \Big\} \tag{3.6}$$

is the metaplectic group associated with $\mathcal{M}$ that is a two-fold covering group of G. The restriction $R_{2,\mathcal{M}}$ of $\widetilde{R}_{\mathcal{M}}$ to $G_{2,\mathcal{M}}$ is the Weil representation of G associated with $\mathcal{M}$. Now we define the projective representation $\pi_{\mathcal{M}}$ of the Jacobi group G^J by

$$\pi_{\mathcal{M}}(hg) = \mathscr{W}_{\mathcal{M}}(h)\, R_{\mathcal{M}}(g), \quad h \in H_{\mathbb{R}}^{(n,m)},\ g \in G. \tag{3.7}$$

Here we identified $h = (\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)}$ (resp. $g \in Sp(n,\mathbb{R})$) with $(I_{2n},(\lambda,\mu;\kappa)) \in G^J$ (resp. $(g,(0,0;0)) \in G^J$). The projective representation $\pi_{\mathcal{M}}$ of G^J is naturally extended to the true representation $\omega_{\mathcal{M}}$ of the group $G^J_{2,\mathcal{M}} = G_{2,\mathcal{M}} \ltimes H_{\mathbb{R}}^{(n,m)}$. The representation $\omega_{\mathcal{M}}$ is called the Schrödinger-Weil representation of G^J. Indeed we have

$$\omega_{\mathcal{M}}(h\cdot(g,t)) = t^m\, \pi_{\mathcal{M}}(hg) = t^m\, \mathscr{W}_{\mathcal{M}}(h)\, R_{\mathcal{M}}(g), \quad h \in H_{\mathbb{R}}^{(n,m)},\ (g,t) \in G_{2,\mathcal{M}}. \tag{3.8}$$

Here we identified $h = (\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)}$ (resp. $(g,t) \in G_{2,\mathcal{M}}$) with $((I_{2n},1),(\lambda,\mu;\kappa)) \in G^J_{2,\mathcal{M}}$ (resp. $((g,t),(0,0;0)) \in G^J_{2,\mathcal{M}}$).

We recall that the following matrices

$$\begin{aligned} t(b) &= \begin{pmatrix} I_n & b \\ 0 & I_n \end{pmatrix} \text{ with any } b = {}^t b \in \mathbb{R}^{(n,n)}, \\ g(\alpha) &= \begin{pmatrix} {}^t\alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \text{ with any } \alpha \in GL(n,\mathbb{R}), \\ \sigma_n &= \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \end{aligned}$$

generate the symplectic group $G = Sp(n,\mathbb{R})$ (cf. [5, p. 326], [13, p. 210]). Therefore the following elements $h_t(\lambda,\mu;\kappa)$, $t_{\mathcal{M}}(b;t)$, $g_{\mathcal{M}}(\alpha;t)$ and $\sigma_{n,\mathcal{M};t}$ of $G_{\mathcal{M}} \ltimes H_{\mathbb{R}}^{(n,m)}$ defined by

$$\begin{aligned}
&h_t(\lambda,\mu;\kappa) = \big((I_{2n},t),(\lambda,\mu;\kappa)\big) \text{ with } t \in T,\ \lambda,\mu \in \mathbb{R}^{(m,n)} \text{ and } \kappa \in \mathbb{R}^{(m,m)},\\
&t_{\mathcal{M}}(b;t) = \big((t(b),t),(0,0;0)\big) \text{ with any } b = {}^t b \in \mathbb{R}^{(n,n)},\ t \in T,\\
&g_{\mathcal{M}}(\alpha;t) = \big((g(\alpha),t),(0,0;0)\big) \text{ with any } \alpha \in GL(n,\mathbb{R}) \text{ and } t \in T,\\
&\sigma_{n,\mathcal{M};t} = ((\sigma_n,t),(0,0;0)) \text{ with } t \in T
\end{aligned}$$

generate the group $G_{\mathcal{M}} \ltimes H_{\mathbb{R}}^{(n,m)}$. We can show that the representation $\widetilde{R}_{\mathcal{M}}$ is realized on the representation $H(\chi_{\mathcal{M}}) = L^2\big(\mathbb{R}^{(m,n)}\big)$ as follows: for each $f \in L^2\big(\mathbb{R}^{(m,n)}\big)$ and $x \in \mathbb{R}^{(m,n)}$, the actions of $\widetilde{R}_{\mathcal{M}}$ on the generators are given by

$$\Big(\widetilde{R}_{\mathcal{M}}\big(h_t(\lambda,\mu;\kappa)\big)f\Big)(x) = t^m\, e^{\pi i\,\sigma\{\mathcal{M}(\kappa+\mu\,{}^t\lambda+2\,x\,{}^t\mu)\}}\, f(x+\lambda), \tag{3.9}$$

$$\Big(\widetilde{R}_{\mathcal{M}}\big(t_{\mathcal{M}}(b;t)\big)f\Big)(x) = t^m\, e^{\pi i\,\sigma(\mathcal{M}\,x\,b\,{}^t x)} f(x), \tag{3.10}$$

$$\Big(\widetilde{R}_{\mathcal{M}}\big(g_{\mathcal{M}}(\alpha;t)\big)f\Big)(x) = t^m\,\big(\det\alpha\big)^{\frac{m}{2}}\, f(x\,{}^t\alpha), \tag{3.11}$$

$$\Big(\widetilde{R}_{\mathcal{M}}\big(\sigma_{n,\mathcal{M};t}\big)f\Big)(x) = t^m \left(\frac{1}{i}\right)^{\frac{mn}{2}} \big(\det\mathcal{M}\big)^{\frac{n}{2}} \int_{\mathbb{R}^{(m,n)}} f(y)\, e^{-2\,\pi i\,\sigma(\mathcal{M}\,y\,{}^t x)}\, dy. \tag{3.12}$$

We denote by $L^2_+\big(\mathbb{R}^{(m,n)}\big)$ $\big($resp. $L^2_-\big(\mathbb{R}^{(m,n)}\big)\big)$ the subspace of $L^2\big(\mathbb{R}^{(m,n)}\big)$ consisting of even (resp. odd) functions in $L^2\big(\mathbb{R}^{(m,n)}\big)$. According to Formulas (3.10)–(3.12), $R_{2,\mathcal{M}}$ is decomposed into representations of $R^{\pm}_{2,\mathcal{M}}$

$$R_{2,\mathcal{M}} = R^+_{2,\mathcal{M}} \oplus R^-_{2,\mathcal{M}},$$

where $R^+_{2,\mathcal{M}}$ and $R^-_{2,\mathcal{M}}$ are the even Weil representation and the odd Weil representation of G that are realized on $L^2_+\big(\mathbb{R}^{(m,n)}\big)$ and $L^2_-\big(\mathbb{R}^{(m,n)}\big)$ respectively. Obviously the center $\mathscr{Z}^J_{2,\mathcal{M}}$ of $G^J_{2,\mathcal{M}}$ is given by

$$\mathscr{Z}^J_{2,\mathcal{M}} = \big\{\big((I_{2n},1),(0,0;\kappa)\big) \in G^J_{2,\mathcal{M}}\big\} \cong S(m,\mathbb{R}).$$

We note that the restriction of $\omega_{\mathcal{M}}$ to $G_{2,\mathcal{M}}$ coincides with $R_{2,\mathcal{M}}$ and $\omega_{\mathcal{M}}(h) = \mathscr{W}_{\mathcal{M}}(h)$ for all $h \in H_{\mathbb{R}}^{(n,m)}$.

Remark 3.1. *In the case $n = m = 1$, $\omega_{\mathcal{M}}$ is dealt in* [1] *and* [12]. *We refer to* [6] *and* [9] *for more details about the Weil representation $R_{2,\mathcal{M}}$.*

Remark 3.2. *The Schrödinger-Weil representation is applied usefully to the theory of Jacobi's sum* [12] *and the theory of Maass-Jacobi forms* [14].

4. Jacobi Forms of Half Integral Weight

Let ρ be a rational representation of $GL(n,\mathbb{C})$ on a finite dimensional complex vector space V_ρ. Let $\mathcal{M} \in \mathbb{R}^{(m,m)}$ be a symmetric half-integral semi-positive definite matrix of degree m. Let $C^\infty(\mathbb{H}_{n,m}, V_\rho)$ be the algebra of all C^∞ functions on $\mathbb{H}_{n,m}$ with values in V_ρ. For $f \in C^\infty(\mathbb{H}_{n,m}, V_\rho)$, we define

$$
\begin{aligned}
&(f|_{\rho,\mathcal{M}}[(g,(\lambda,\mu;\kappa))])(\Omega,Z)\\
&:= e^{-2\pi i\,\sigma\left(\mathcal{M}(Z+\lambda\,\Omega+\mu)(C\Omega+D)^{-1}C\,{}^t(Z+\lambda\,\Omega+\mu)\right)} \times e^{2\pi i\,\sigma\left(\mathcal{M}(\lambda\,\Omega\,{}^t\lambda+2\,\lambda\,{}^tZ+\kappa+\mu\,{}^t\lambda)\right)}\\
&\quad\times \rho(C\,\Omega+D)^{-1} f(g\cdot\Omega,(Z+\lambda\,\Omega+\mu)(C\,\Omega+D)^{-1}),
\end{aligned}
\tag{4.1}
$$

where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R}),\ (\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)}$ and $(\Omega, Z) \in \mathbb{H}_{n,m}$.

Definition 4.1. *Let ρ and $\mathcal{M}$ be as above. Let*

$$H_{\mathbb{Z}}^{(n,m)} := \{(\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)} \mid \lambda,\mu \in \mathbb{Z}^{(m,n)},\ \kappa \in \mathbb{Z}^{(m,m)}\ \}.$$

A Jacobi form of index $\mathcal{M}$ with respect to ρ on a subgroup Γ of Γ_n of finite index is a holomorphic function $f \in C^\infty(\mathbb{H}_{n,m}, V_\rho)$ satisfying the following conditions (A) and (B):

(A) $f|_{\rho,\mathcal{M}}[\tilde\gamma] = f$ for all $\tilde\gamma \in \widetilde\Gamma := \Gamma \ltimes H_{\mathbb{Z}}^{(n,m)}$.

(B) For each $M \in \Gamma_n$, $f|_{\rho,\mathcal{M}}$ has a Fourier expansion of the following form :

$$f(\Omega,Z) = \sum_{\substack{T={}^tT\geq 0\\ \text{half-integral}}} \sum_{R\in\mathbb{Z}^{(n,m)}} c(T,R)\cdot e^{\frac{2\pi i}{\lambda_\Gamma}\,\sigma(T\Omega)} \cdot e^{2\pi i\,\sigma(RZ)}$$

with a suitable $\lambda_\Gamma \in \mathbb{Z}^\times$ and $c(T,R) \neq 0$ only if $\begin{pmatrix} \frac{1}{\lambda_\Gamma}T & \frac12 R \\ \frac12 {}^tR & \mathcal{M} \end{pmatrix} \geqq 0$.

If $n \geq 2$, the condition (B) is superfluous by Köcher principle (cf. [39] Lemma 1.6). We denote by $J_{\rho,\mathcal{M}}(\Gamma)$ the vector space of all Jacobi forms of index $\mathcal{M}$ with respect to ρ on Γ. Ziegler (cf. [39] Theorem 1.8 or [4] Theorem 1.1) proves that the vector space $J_{\rho,\mathcal{M}}(\Gamma)$ is finite dimensional. In the special case $\rho(A) = (\det(A))^k$ with $A \in GL(n,\mathbb{C})$ and a fixed $k \in \mathbb{Z}$, we write $J_{k,\mathcal{M}}(\Gamma)$ instead of $J_{\rho,\mathcal{M}}(\Gamma)$ and call k the *weight* of the corresponding Jacobi forms. For more results on Jacobi forms with $n > 1$ and $m > 1$, we refer to [29]-[32] and [39]. Jacobi forms play an important role in lifting elliptic cusp forms to Siegel cusp forms of degree $2n$ (cf. [8]).

Definition 4.2. *A Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma)$ is said to be a cusp (or cuspidal) form if $\begin{pmatrix} \frac{1}{\lambda_\Gamma}T & \frac12 R \\ \frac12 {}^tR & \mathcal{M} \end{pmatrix} > 0$ for any T, R with $c(T,R) \neq 0$. A Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma)$ is said to be singular if it admits a Fourier expansion such that a Fourier coefficient $c(T,R)$ vanishes unless det $\begin{pmatrix} \frac{1}{\lambda_\Gamma}T & \frac12 R \\ \frac12 {}^tR & \mathcal{M} \end{pmatrix} = 0$.*

Remark 4.1. *Singular Jacobi forms were characterized by a certain differential operator and the weight by the author* [31].

Without loss of generality we may assume that ρ is irreducible. Then we choose a hermitian inner product $\langle\ ,\ \rangle$ on V_ρ that is preserved under the unitary group $U(n) \subset$

$GL(n,\mathbb{C})$. For two Jacobi forms f_1 and f_2 in $J_{\rho,\mathcal{M}}(\Gamma)$, we define the Petersson inner product formally by

$$(f_1,f_2):=\int_{\Gamma_{n,m}\backslash\mathbb{H}_{n,m}}\langle\,\rho(Y^{\frac12})f_1(\Omega,Z),\rho(Y^{\frac12})f_2(\Omega,Z)\rangle\,\kappa_{\mathcal{M}}(\Omega,Z)\,dv. \tag{4.2}$$

Here

$$dv=(\det Y)^{-(n+m+1)}[dX]\wedge[dY]\wedge[dU]\wedge[dV] \tag{4.3}$$

is a G^J-invariant volume element on $\mathbb{H}_{n,m}$ and

$$\kappa_{\mathcal{M}}(\Omega,Z):=e^{-4\pi i\,\sigma(\,{}^t(\mathrm{Im}\,Z)\,\mathcal{M}\,\mathrm{Im}\,Z\,(\mathrm{Im}\,\Omega)^{-1})}=e^{-4\pi i\,\sigma(\,{}^tV\mathcal{M}VY^{-1})}, \tag{4.4}$$

where $\Omega=X+i\,Y\in\mathbb{H}_n,\ Z=U+i\,V\in\mathbb{C}^{(m,n)},\ X=(x_{ij}),\ Y=(y_{ij}),\ U=(u_{kl}),\ V=(v_{kl})$ real and

$$[dX]=\bigwedge_{i\le j}dx_{ij},\quad [dY]=\bigwedge_{i\le j}dy_{ij},\quad [dU]=\bigwedge_{k\le l}du_{kl}\quad\text{and}\quad [dV]=\bigwedge_{k\le l}dv_{kl}.$$

A Jacobi form f in $J_{\rho,\mathcal{M}}(\Gamma)$ is said to be *square integrable* if $(f,f)<\infty$. We note that cusp Jacobi forms are square integrable and that (f_1,f_2) is finite if one of f_1 and f_2 is a cusp Jacobi form (cf. [39], p. 203).

For $g=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in G$, we set

$$J(g,\Omega)=C\Omega+D,\quad \Omega\in\mathbb{H}_n. \tag{4.5}$$

We define the map $J_{\mathcal{M}}:G^J\times\mathbb{H}_{n,m}\longrightarrow\mathbb{C}^\times$ by

$$J_{\mathcal{M}}\big(\widetilde{g},(\Omega,Z)\big):=e^{2\pi i\,\sigma(\mathcal{M}[Z+\lambda\Omega+\mu](C\Omega+D)^{-1}C)}\cdot e^{-2\pi i\,\sigma(\mathcal{M}(\lambda\,\Omega\,{}^t\lambda\,+\,2\lambda\,{}^tZ+\kappa+\mu\,{}^t\lambda))}, \tag{4.6}$$

where $\widetilde{g}=(g,(\lambda,\mu;\kappa))\in G^J$ with $g=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in G$ and $(\lambda,\mu;\kappa)\in H_{\mathbb{R}}^{(n,m)}$. Here we use the Siegel's notation $S[X]:={}^tXSX$ for two matrices S and X.

We define the map $J_{\rho,\mathcal{M}}:G^J\times\mathbb{H}_{n,m}\longrightarrow GL(V_\rho)$ by

$$J_{\rho,\mathcal{M}}(\widetilde{g},(\Omega,Z))=J_{\mathcal{M}}(\widetilde{g},(\Omega,Z))\,\rho(J(g,\Omega)), \tag{4.7}$$

where $\widetilde{g}=(g,h)\in G^J$ with $g\in G$ and $h\in H_{\mathbb{R}}^{(n,m)}$. For a function f on $\mathbb{H}_n$ with values in V_ρ, we can lift f to a function Φ_f on G^J :

$$\begin{aligned}\Phi_f(\sigma):\ &=\ (f|_{\rho,\mathcal{M}}[\sigma])(iI_n,0)\\ &=\ J_{\rho,\mathcal{M}}(\sigma,(iI_n,0))^{-1}f(\sigma\cdot(iI_n,0)),\quad \sigma\in G^J.\end{aligned}$$

A characterization of Φ_f for a cusp Jacobi form f in $J_{\rho,\mathcal{M}}(\Gamma)$ was given by Takase [19, pp. 162–164].

We allow a weight k to be half-integral. Let

$$\mathfrak{S}=\Big\{\,S\in\mathbb{C}^{(n,n)}\mid S={}^tS,\ \mathrm{Re}\,(S)>0\,\Big\}$$

be a connected simply connected complex manifold. Then there is a uniquely determined holomorphic function $\det^{1/2}$ on $\mathfrak{S}$ such that

$$\left(\det^{1/2} S\right)^2 = \det S \qquad \text{for all } S \in \mathfrak{S}, \tag{4.8}$$

$$\det^{1/2} S = (\det S)^{1/2} \qquad \text{for all } S \in \mathfrak{S} \cap \mathbb{R}^{(n,n)}. \tag{4.9}$$

For each integer $k \in \mathbb{Z}$ and $S \in \mathfrak{S}$, we put

$$\det^{k/2} S = \left(\det^{1/2} S\right)^k.$$

For brevity, we set $G = Sp(n, \mathbb{R})$. For any $g \in G$ and $\Omega, \Omega' \in \mathbb{H}_n$, we put

$$\begin{aligned} \varepsilon(g; \Omega', \Omega) &= \det^{-1/2}\left(\frac{g\cdot\Omega' - \overline{g\cdot\Omega}}{2\,i}\right) \det^{1/2}\left(\frac{\Omega' - \overline{\Omega}}{2\,i}\right) \\ &\quad \times |\det J(g, \Omega')|^{-1/2}\, |\det J(g, \Omega)|^{-1/2}. \end{aligned} \tag{4.10}$$

For each $\Omega \in \mathbb{H}_n$, we define the function $\beta_\Omega : G \times G \longrightarrow T$ by

$$\beta_\Omega(g_1, g_2) = \epsilon(g_1; \Omega, g_2(\Omega)), \quad g_1, g_2 \in G. \tag{4.11}$$

Then β_Ω satisfies the cocycle condition and the cohomology class of β_Ω of order two:

$$\beta_\Omega(g_1, g_2)^2 = \alpha_\Omega(g_2)\, \alpha_\Omega(g_1 g_2)^{-1}\, \alpha_\Omega(g_1), \tag{4.12}$$

where

$$\alpha_\Omega(g) = \frac{\det J(g, \Omega)}{|\det J(g, \Omega)|}, \quad g \in G,\ \Omega \in \mathbb{H}_n. \tag{4.13}$$

For any $\Omega \in \mathbb{H}_n$, we let

$$G_\Omega = \left\{ (g, \epsilon) \in G \times T \mid \epsilon^2 = \alpha_\Omega(g)^{-1} \right\}$$

be the two-fold covering group with the multiplication law

$$(g_1, \epsilon_1)(g_2, \epsilon_2) = \big(g_1 g_2,\ \epsilon_1 \epsilon_2 \beta_\Omega(g_1, g_2)\big).$$

The covering group G_Ω depends on the choice of $\Omega \in \mathbb{H}_n$, i.e., the choice of a maximal compact subgroup of G. However for any two element $\Omega_1, \Omega_2 \in \mathbb{H}_n$, G_{Ω_1} is isomorphic to G_{Ω_2} (cf. [21]). We put

$$G_* := G_{iI_n}. \tag{4.14}$$

Takase [21, p. 131] defined the automorphic factor $J_{1/2} : G_* \times \mathbb{H}_n \longrightarrow \mathbb{C}^\times$ by

$$J_{1/2}(g_\epsilon, \Omega) := \epsilon^{-1} \varepsilon(g; \Omega, iI_n) |\det J(g, \Omega)|^{1/2}, \tag{4.15}$$

where $g_\epsilon = (g, \epsilon) \in G_*$ with $g \in G$ and $\Omega \in \mathbb{H}_n$. It is easily checked that

$$J_{1/2}(g_* h_*, \Omega) = J_{1/2}(g_*, h\cdot\Omega) J_{1/2}(h_*, \Omega) \tag{4.16}$$

for all $g_* = (g, \epsilon)$, $h_* = (h, \eta) \in G_*$ and $\Omega \in \mathbb{H}_n$. Moreover

$$J_{1/2}(g_*, \Omega)^2 = \det(C\Omega + D) \tag{4.17}$$

for all $g_* = (g, \epsilon) \in G_*$ with $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$.

Let $\pi_* : G_* \longrightarrow G$ be the projection defined by $\pi_*(g, \epsilon) = g$. Let Γ be a subgroup of the Siegel modular group Γ_n of finite index. Let $\Gamma_* = \pi_*^{-1}(\Gamma) \subset G_*$. Let χ be a finite order unitary character of Γ_*. Let $k \in \mathbb{Z}^+$ be a positive integer. We say that a holomorphic function $\phi : \mathbb{H}_n \longrightarrow \mathbb{C}$ is a Siegel modular form of a half-integral weight $k/2$ with level Γ if it satisfies the condition

$$\phi(\gamma_* \cdot \Omega) = \chi(\gamma_*) J_{1/2}(\gamma_*, \Omega)^k \phi(\Omega) \tag{4.18}$$

for all $\gamma_* \in \Gamma_*$ and $\Omega \in \mathbb{H}_n$. We denote by $M_{k/2}(\Gamma, \chi)$ be the vector space of all Siegel modular forms of weight $k/2$ with level Γ. Let $S_{k/2}(\Gamma, \chi)$ be the subspace of $M_{k/2}(\Gamma, \chi)$ consisting of $\phi \in M_{k/2}(\Gamma, \chi)$ such that

$$|\phi(\Omega)| \det(\operatorname{Im} \Omega)^{k/4} \text{ is bounded on } \mathbb{H}_n.$$

An element of $S_{k/2}(\Gamma, \chi)$ is called a Siegel cusp form of weight $k/2$. The Petersson norm on $S_{k/2}(\Gamma, \chi)$ is defined by

$$||\phi||^2 = \int_{\Gamma \backslash \mathbb{H}_n} |\phi(\Omega)|^2 \det(\operatorname{Im} \Omega)^{k/2} \, dv_\Omega,$$

where

$$dv_\Omega = (\det Y)^{-(n+1)} [dX][dY]$$

is a G-invariant volume element on $\mathbb{H}_n$.

Remark 4.2. *Using the Schrödinger-Weil representation, Takase* [22] *established a bijective correspondence between the space of cuspidal Jacobi forms and the space of Siegel cusp forms of half integral weight which is compatible with the action of Hecke operators. For example, if m is a positive integer, the classical result (cf.* [4] *and* [7]*)*

$$J_{m,1}^{cusp}(\Gamma_n) \cong S_{m-1/2}(\Gamma_0(4))$$

can be obtained by the method of the representation theory. Here $\Gamma_0(4)$ is the Hecke subgroup of the Siegel modular group Γ_n and $J_{m,1}^{cusp}(\Gamma_n)$ denotes the vector space of cuspidal Jacobi forms of weight m and index 1.

Now we are in a position to define the notion of Jacobi forms of half integral weight as follows.

Definition 4.3. *Let $\Gamma \subset \Gamma_n$ be a subgroup of finite index. We put $\Gamma_* = \pi_*^{-1}(\Gamma)$ and*

$$\Gamma_*^J = \Gamma_* \ltimes H_{\mathbb{Z}}^{(n,m)}.$$

A holomorphic function $f : \mathbb{H}_{n,m} \longrightarrow \mathbb{C}$ is said to be a Jacobi form of a weight $k/2 \in \frac{1}{2}\mathbb{Z}$ (k: odd) with level Γ and index $\mathcal{M}$ for the character χ of Γ_*^J of if it satisfies the following transformation formula

$$f(\widetilde{\gamma}_* \cdot (\Omega, Z)) = \chi(\gamma_*) \, J_{k,\mathcal{M}}(\widetilde{\gamma}_*, (\Omega, Z)) f(\Omega, Z) \quad \text{for all } \widetilde{\gamma}_* \in \Gamma_*^J. \tag{4.19}$$

Here $J_{k,\mathcal{M}} : \Gamma_*^J \times \mathbb{H}_{n,m} \longrightarrow \mathbb{C}$ is an automorphic factor defined by

$$\begin{aligned} J_{k,\mathcal{M}}\big(\widetilde{\gamma}_*, (\Omega, Z)\big) : \; &= \; e^{2\pi i \sigma\left(\mathcal{M}(Z+\lambda\Omega+\mu)(C\Omega+D)^{-1}C\,{}^t(Z+\lambda\Omega+\mu)\right)} \\ &\quad \times e^{-2\pi i\sigma\left(\mathcal{M}(\lambda\,\Omega\,{}^t\lambda + 2\lambda\,{}^tZ + \kappa + \mu\,{}^t\lambda)\right)} J_{1/2}(\gamma_*, \Omega)^k, \end{aligned} \tag{4.20}$$

where $\widetilde{\gamma}_* = (\gamma_*, (\lambda, \mu; \kappa)) \in \Gamma_*^J$ *with* $\gamma_* = (\gamma, \epsilon) \in \Gamma_*,\ \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma,\ (\lambda, \mu, \kappa) \in H_{\mathbb{Z}}^{(n,m)}$ *and* $(\Omega, Z) \in \mathbb{H}_{n,m}$.

5. Covariant Maps for the Schrödinger-Weil representation

As before we let $\mathcal{M}$ be a symmetric positive definite $m \times m$ real matrix. We keep the notations in the previous sections.

We define the mapping $\mathscr{F}^{(\mathcal{M})} : \mathbb{H}_{n,m} \longrightarrow L^2\big(\mathbb{R}^{(m,n)}\big)$ by

$$\mathscr{F}^{(\mathcal{M})}(\Omega, Z)(x) = e^{\pi i\, \sigma\{\mathcal{M}(x\,\Omega\,{}^t x + 2\,x\,{}^t Z)\}}, \quad (\Omega, Z) \in \mathbb{H}_{n,m},\ x \in \mathbb{R}^{(m,n)}. \tag{5.1}$$

For brevity we put $\mathscr{F}_{\Omega,Z}^{(\mathcal{M})} := \mathscr{F}^{(\mathcal{M})}(\Omega, Z)$ for $(\Omega, Z) \in \mathbb{H}_{n,m}$. Takase [21] proved that $G_{2,\mathcal{M}}$ is isomorphic to G_{iI_n} (cf. (3.6) and (4.14)). Therefore we will use $G_* := G_{iI_n}$ instead of $G_{2,\mathcal{M}}$.

We set

$$G_*^J := G_* \ltimes H_{\mathbb{R}}^{(n,m)}. \tag{5.2}$$

We note that G_*^J acts on $\mathbb{H}_{n,m}$ via the canonical projection of G_*^J onto G^J.

Now we assume that m is odd. We define the automorphic factor $J_{\mathcal{M}}^* : G_*^J \times \mathbb{H}_{n,m} \longrightarrow \mathbb{C}^\times$ for G_*^J on $\mathbb{H}_{n,m}$ by

$$\begin{aligned} J_{\mathcal{M}}^*(\widetilde{g}_*, (\Omega, Z)) &= e^{\pi i\, \sigma\left(\mathcal{M}(Z+\lambda\,\Omega+\mu)(C\Omega+D)^{-1}C\,{}^t(Z+\lambda\,\Omega+\mu)\right)} \\ &\quad \times e^{-\pi i\, \sigma\left(\mathcal{M}(\lambda\,\Omega\,{}^t\lambda + 2\,\lambda\,{}^t Z + \kappa + \mu\,{}^t\lambda)\right)} J_{1/2}((g, \epsilon), \Omega)^m, \end{aligned} \tag{5.3}$$

where $\widetilde{g}_* = ((g, \epsilon), (\lambda, \mu; \kappa)) \in G_*^J$ with $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G,\ (\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(n,m)}$ and $(\Omega, Z) \in \mathbb{H}_{n,m}$.

Theorem 5.1. *Let m be an odd positive integer. Then the map $\mathscr{F}^{(\mathcal{M})} : \mathbb{H}_{n,m} \longrightarrow L^2\big(\mathbb{R}^{(m,n)}\big)$ defined by (5.1) is a* covariant map *for the Schrödinger-Weil representation $\omega_{\mathcal{M}}$ of G_*^J and the automorphic factor $J_{\mathcal{M}}^*$ for G_*^J on $\mathbb{H}_{n,m}$ defined by Formula (5.3). In other words, $\mathscr{F}^{(\mathcal{M})}$ satisfies the following covariance relation*

$$\omega_{\mathcal{M}}(\widetilde{g}_*)\mathscr{F}_{\Omega,Z}^{(\mathcal{M})} = J_{\mathcal{M}}^*\big(\widetilde{g}_*, (\Omega, Z)\big)^{-1} \mathscr{F}_{\widetilde{g}_* \cdot (\Omega, Z)}^{(\mathcal{M})} \tag{5.4}$$

for all $\widetilde{g}_* \in G_*^J$ and $(\Omega, Z) \in \mathbb{H}_{n,m}$.

Proof. For an element $\widetilde{g}_* = ((g, \epsilon), (\lambda, \mu; \kappa)) \in G_*^J$ with $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$, we put $(\Omega_*, Z_*) = \widetilde{g}_* \cdot (\Omega, Z)$ for $(\Omega, Z) \in \mathbb{H}_{n,m}$. Then we have

$$\begin{aligned} \Omega_* &= g \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1}, \\ Z_* &= (Z + \lambda\,\Omega + \mu)(C\Omega + D)^{-1}. \end{aligned}$$

In this section we use the notations $t(b)$, $g(\alpha)$ and σ_n in Section 3. Since the following elements $h(\lambda,\mu;\kappa)_\epsilon$, $t(b;\epsilon)$, $g(\alpha,\epsilon)$ and $\sigma_{n,\epsilon}$ of G_*^J defined by

$$\begin{aligned} h(\lambda,\mu;\kappa)_\epsilon &= \big((I_{2n},\epsilon),(\lambda,\mu;\kappa)\big) \quad \text{with } \epsilon=\pm 1,\ \lambda,\mu\in\mathbb{R}^{(m,n)},\ \kappa\in\mathbb{R}^{(m,m)}, \\ t(b;\epsilon) &= \big((t(b),\epsilon),(0,0;0)\big) \quad \text{with } \epsilon=\pm 1,\ b\in S(n,\mathbb{R}), \\ g(\alpha;\epsilon) &= \big((g(\alpha),\epsilon),(0,0;0)\big) \quad \text{with } \epsilon=\pm 1 \text{ if } \det\alpha>0,\ \epsilon=\pm i \text{ if } \det\alpha<0, \\ \sigma_{n,\epsilon} &= \big((\sigma_n,\epsilon),(0,0;0)\big) \text{ with } \epsilon^2=(-i)^n. \end{aligned}$$

generate the group G_*^J, it suffices to prove the covariance relation (5.4) for the above generators.

Case I. $\widetilde{g}_* = h(\lambda,\mu;\kappa)_\epsilon$ with $\epsilon=\pm 1$ and $\lambda,\mu\in\mathbb{R}^{(m,n)}$, $\kappa\in\mathbb{R}^{(m,m)}$.

In this case, we have

$$\Omega_*=\Omega,\quad Z_*=Z+\lambda\,\Omega+\mu.$$

We put

$$h(\lambda,\mu;\kappa)_+ := \big((I_{2n},1),(\lambda,\mu;\kappa)\big)$$

and

$$h(\lambda,\mu;\kappa)_- := \big((I_{2n},-1),(\lambda,\mu;\kappa)\big).$$

It is easily seen that according to Formula (4.15), we obtain

$$J_{1/2}((I_{2n},1),\Omega)=1 \quad\text{and}\quad J_{1/2}((I_{2n},-1),\Omega)=-1.$$

Therefore we get

$$J_{\mathcal{M}}^*\big(h(\lambda,\mu;\kappa)_+,(\Omega,Z)\big) = e^{-\pi i\,\sigma\{\mathcal{M}(\lambda\,\Omega\,{}^t\lambda+2\,\lambda\,{}^tZ+\kappa+\mu\,{}^t\lambda)\}}$$

and

$$J_{\mathcal{M}}^*\big(h(\lambda,\mu;\kappa)_-,(\Omega,Z)\big) = -e^{-\pi i\,\sigma\{\mathcal{M}(\lambda\,\Omega\,{}^t\lambda+2\,\lambda\,{}^tZ+\kappa+\mu\,{}^t\lambda)\}}.$$

According to Formulas (2.4), (3.8) and (3.9), for $x\in\mathbb{R}^{(m,n)}$,

$$\begin{aligned} &\Big(\omega_{\mathcal{M}}\big(h(\lambda,\mu;\kappa)_+\big)\mathscr{F}_{\Omega,Z}^{(\mathcal{M})}\Big)(x) \\ =\ & e^{\pi i\,\sigma\{\mathcal{M}(\kappa+\mu\,{}^t\lambda+2\,x\,{}^t\mu)\}}\,\mathscr{F}_{\Omega,Z}^{(\mathcal{M})}(x+\lambda) \\ =\ & e^{\pi i\,\sigma\{\mathcal{M}(\kappa+\mu\,{}^t\lambda+2\,x\,{}^t\mu)\}}\,e^{\pi i\,\sigma\{\mathcal{M}((x+\lambda)\Omega\,{}^t(x+\lambda)+\,2\,(x+\lambda)\,{}^tZ)\}}. \end{aligned}$$

On the other hand, according to Formula (5.3), for $x\in\mathbb{R}^{(m,n)}$,

$$\begin{aligned} &J_{\mathcal{M}}^*\big(h(\lambda,\mu;\kappa)_+,(\Omega,Z)\big)^{-1}\mathscr{F}_{\widetilde{g}_*\cdot(\Omega,Z)}^{(\mathcal{M})}(x) \\ =\ & J_{\mathcal{M}}^*\big(h(\lambda,\mu;\kappa)_+,(\Omega,Z)\big)^{-1}\mathscr{F}_{\Omega,Z+\lambda\,\Omega+\mu}^{(\mathcal{M})}(x) \\ =\ & e^{\pi i\,\sigma\{\mathcal{M}(\lambda\Omega\,{}^t\lambda\,+\,2\,\lambda\,{}^tZ\,+\,\kappa\,+\,\mu\,{}^t\lambda)\}}\cdot e^{\pi i\,\sigma\{\mathcal{M}(x\,\Omega\,{}^tx\,+\,2\,x\,{}^t(Z+\lambda\Omega+\mu))\}} \\ =\ & e^{\pi i\,\sigma\{\mathcal{M}(\kappa+\mu\,{}^t\lambda+2\,x\,{}^t\mu)\}}\,e^{\pi i\,\sigma\{\mathcal{M}((x+\lambda)\Omega\,{}^t(x+\lambda)+\,2\,(x+\lambda)\,{}^tZ)\}}. \end{aligned}$$

Therefore we prove the covariance relation (5.4) in the case $\widetilde{g}_* = h(\lambda,\mu;\kappa)_+$.

Similarly we can prove the covariance relation (5.4) in the $\widetilde{g}_* = h(\lambda,\mu;\kappa)_-$. In fact,

$$\begin{aligned} & \left(\omega_{\mathcal{M}}\big(h(\lambda,\mu;\kappa)_-\big)\mathscr{F}^{(\mathcal{M})}_{\Omega,Z}\right)(x) \\ = \;& -e^{\pi i\,\sigma\{\mathcal{M}(\kappa+\mu\,{}^t\lambda+2\,x\,{}^t\mu)\}}\,e^{\pi i\,\sigma\{\mathcal{M}((x+\lambda)\Omega\,{}^t(x+\lambda)+2\,(x+\lambda)\,{}^tZ)\}} \\ = \;& J^*_{\mathcal{M}}\big(h(\lambda,\mu;\kappa)_-,(\Omega,Z)\big)^{-1}\mathscr{F}^{(\mathcal{M})}_{\widetilde{g}_*\cdot(\Omega,Z)}(x). \end{aligned}$$

Case II. $\widetilde{g}_* = t(b;\epsilon)$ with $b \in S(n,\mathbb{R})$.

In this case, we have

$$\Omega_* = \Omega + b \quad \text{and} \quad Z_* = Z.$$

We put

$$t(b)_+ = \big((t(b),1),(0,0;0)\big)$$

and

$$t(b)_- = \big((t(b),-1),(0,0;0)\big).$$

It is easily seen that

$$J_{1/2}((t(b),1),\Omega) = 1 \quad \text{and} \quad J_{1/2}((t(b),-1),\Omega) = -1.$$

Therefore we get

$$J^*_{\mathcal{M}}\big(t(b)_+,(\Omega,Z)\big) = 1 \quad \text{and} \quad J^*_{\mathcal{M}}\big(t(b)_-,(\Omega,Z)\big) = -1.$$

According to Formulas (3.8) and (3.10), we obtain

$$\left(\omega_{\mathcal{M}}\big(t(b)_+\big)\mathscr{F}^{(\mathcal{M})}_{\Omega,Z}\right)(x) = e^{\pi\,i\,\sigma(\mathcal{M}\,xb\,{}^tx)}\,\mathscr{F}^{(\mathcal{M})}_{\Omega,Z}(x), \quad x \in \mathbb{R}^{(m,n)}.$$

On the other hand, according to Formula (5.3), for $x \in \mathbb{R}^{(m,n)}$, we obtain

$$\begin{aligned} & J^*_{\mathcal{M}}\big(t(b)_+,(\Omega,Z)\big)^{-1}\mathscr{F}^{(\mathcal{M})}_{t(b)_+\cdot(\Omega,Z)}(x) \\ = \;& \mathscr{F}^{(\mathcal{M})}_{\Omega+b,Z}(x) \\ = \;& e^{\pi i\,\sigma\left(\mathcal{M}\left(x(\Omega+b)\,{}^tx+2\,x\,{}^tZ\right)\right)} \\ = \;& e^{\pi\,i\,\sigma(\mathcal{M}\,xb\,{}^tx)}\,\mathscr{F}^{(\mathcal{M})}_{\Omega,Z}(x). \end{aligned}$$

Therefore we prove the covariance relation (5.4) in the case $\widetilde{g}_* = t(b)_+$ with $b \in S(n,\mathbb{R})$.

Similarly we can prove the covariance relation (5.4) in the $\widetilde{g}_* = t(b)_-$. In fact,

$$\begin{aligned} & \left(\omega_{\mathcal{M}}\big(t(b)_-\big)\mathscr{F}^{(\mathcal{M})}_{\Omega,Z}\right)(x) \\ = \;& -e^{\pi\,i\,\sigma(\mathcal{M}\,xb\,{}^tx)}\,\mathscr{F}^{(\mathcal{M})}_{\Omega,Z}(x) \\ = \;& J^*_{\mathcal{M}}\big(t(b)_-,(\Omega,Z)\big)^{-1}\mathscr{F}^{(\mathcal{M})}_{t(b)_-\cdot(\Omega,Z)}(x). \end{aligned}$$

Case III. $\widetilde{g}_* = \big((g(\alpha),\epsilon),(0,0;0)\big)$ with $\epsilon = \pm\,1$ (det $\alpha > 0$) and $\epsilon = \pm\,i$ (det $\alpha < 0$). Here $\alpha \in GL(n,\mathbb{R})$.

In this case, we have

$$\Omega_* = {}^t\alpha\,\Omega\,\alpha \quad \text{and} \quad Z_* = Z\alpha.$$

We put

$$\begin{aligned} g(\alpha)_+ : &= \big((g(\alpha), 1), (0,0;0)\big),\\ g(\alpha)_- : &= \big((g(\alpha), -1), (0,0;0)\big),\\ g(\alpha)^+ : &= \big((g(\alpha), i), (0,0;0)\big),\\ g(\alpha)^- : &= \big((g(\alpha), -i), (0,0;0)\big). \end{aligned}$$

And we can show easily that

$$\begin{aligned} J_{1/2}\big((g(\alpha),1),\Omega\big) &= (\det\,\alpha)^{-1/2},\\ J_{1/2}\big((g(\alpha),-1),\Omega\big) &= -(\det\,\alpha)^{-1/2},\\ J_{1/2}\big((g(\alpha),i),\Omega\big) &= i\,(\det\,\alpha)^{-1/2},\\ J_{1/2}\big((g(\alpha),-i),\Omega\big) &= -i\,(\det\,\alpha)^{-1/2}. \end{aligned}$$

Using Formulas (3.5), (3.11) and (5.3), we can show the covariance relation (5.4) in the case $\widetilde{g}_* = \big((g(\alpha),\epsilon),(0,0;0)\big)$ with $\epsilon = \pm 1,\ \pm i$ and $\alpha \in GL(n,\alpha)$.

Case IV. $\widetilde{g}_* = \big((\sigma_n,\epsilon),(0,0;0)\big)$ with $\sigma_n = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ and $\epsilon^2 = (-i)^n$.

In this case, we have

$$\Omega_* = -\Omega^{-1} \quad \text{and} \quad Z_* = Z\,\Omega^{-1}.$$

In order to prove the covariance relation (5.3), we need the following useful lemma.

Lemma 5.1. *For a fixed element $\Omega \in \mathbb{H}_n$ and a fixed element $Z \in \mathbb{C}^{(m,n)}$, we obtain the following property*

$$\int_{\mathbb{R}^{(m,n)}} e^{\pi\, i\, \sigma(x\,\Omega\, {}^tx + 2\,x\,{}^tZ)} dx_{11}\cdots dx_{mn} = \left(\det \frac{\Omega}{i}\right)^{-\frac{m}{2}} e^{-\pi\, i\, \sigma(Z\,\Omega^{-1}\,{}^tZ)}, \tag{5.5}$$

where $x = (x_{ij}) \in \mathbb{R}^{(m,n)}$.

Proof of Lemma 5.1. By a simple computation, we see that

$$e^{\pi i\,\sigma(x\Omega\,{}^tx + 2x\,{}^tZ)} = e^{-\pi i\,\sigma(Z\Omega^{-1}\,{}^tZ)} \cdot e^{\pi i\,\sigma\{(x+Z\Omega^{-1})\Omega\,{}^t(x+Z\Omega^{-1})\}}.$$

Since the real Jacobi group $Sp(n,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(m,n)}$ acts on $\mathbb{H}_{n,m}$ holomorphically, we may put

$$\Omega = i\,A\,{}^tA, \quad Z = iV, \quad A \in \mathbb{R}^{(n,n)}, \quad V = (v_{ij}) \in \mathbb{R}^{(m,n)}.$$

Then we obtain

$$
\begin{aligned}
& \int_{\mathbb{R}^{(m,n)}} e^{\pi i\,\sigma(x\Omega\,{}^t x+2x\,{}^t Z)} dx_{11}\cdots dx_{mn} \\
= \;& e^{-\pi i\,\sigma(Z\Omega^{-1}\,{}^t Z)} \int_{\mathbb{R}^{(m,n)}} e^{\pi i\,\sigma[\{x+iV(iA\,{}^tA)^{-1}\}(iA\,{}^tA)\,{}^t\{x+iV(iA\,{}^tA)^{-1}\}]}\, dx_{11}\cdots dx_{mn} \\
= \;& e^{-\pi i\,\sigma(Z\Omega^{-1}\,{}^t Z)} \int_{\mathbb{R}^{(m,n)}} e^{\pi i\,\sigma[\{x+V(A\,{}^tA)^{-1}\}A\,{}^tA\,{}^t\{x+V(A\,{}^tA)^{-1}\}]}\, dx_{11}\cdots dx_{mn} \\
= \;& e^{-\pi i\,\sigma(Z\Omega^{-1}\,{}^t Z)} \int_{\mathbb{R}^{(m,n)}} e^{-\pi\,\sigma\{(uA)\,{}^t(uA)\}}\, du_{11}\cdots du_{mn} \quad \big(\text{Put } u = x+V(A\,{}^tA)^{-1} = (u_{ij})\big) \\
= \;& e^{-\pi i\,\sigma(Z\Omega^{-1}\,{}^t Z)} \int_{\mathbb{R}^{(m,n)}} e^{-\pi\,\sigma(w\,{}^tw)} (\det A)^{-m}\, dw_{11}\cdots dw_{mn} \quad \big(\text{Put } w = uA = (w_{ij})\big) \\
= \;& e^{-\pi i\,\sigma(Z\Omega^{-1}\,{}^t Z)}\, (\det A)^{-m}\cdot \left(\prod_{i=1}^{m}\prod_{j=1}^{g} \int_{\mathbb{R}} e^{-\pi\, w_{ij}^2}\, dw_{ij} \right) \\
= \;& e^{-\pi i\,\sigma(Z\Omega^{-1}\,{}^t Z)}\, (\det A)^{-m} \quad \Big(\text{because } \int_{\mathbb{R}} e^{-\pi\, w_{ij}^2}\, dw_{ij} = 1 \quad \text{for all } i,j\Big) \\
= \;& e^{-\pi i\,\sigma(Z\Omega^{-1}\,{}^t Z)}\, \big(\det\big(A\,{}^tA\big)\big)^{-\frac{m}{2}} \\
= \;& e^{-\pi i\,\sigma(Z\Omega^{-1}\,{}^t Z)}\, \left(\det\left(\frac{\Omega}{i}\right)\right)^{-\frac{m}{2}}.
\end{aligned}
$$

This completes the proof of Lemma 5.1. □

According to Formulas (3.8) and (3.12), for $x\in\mathbb{R}^{(m,n)}$, we obtain

$$
\begin{aligned}
& \Big(\omega_{\mathcal{M}}(\widetilde{g}_*)\mathscr{F}^{(\mathcal{M})}_{\Omega,Z}\Big)(x) \\
= \;& \epsilon^m \left(\frac{1}{i}\right)^{\frac{mn}{2}} \big(\det\mathcal{M}\big)^{\frac{n}{2}} \int_{\mathbb{R}^{(m,n)}} \mathscr{F}^{(\mathcal{M})}_{\Omega,Z}(y)\, e^{-2\pi\, i\,\sigma\,(\mathcal{M}\,y\,{}^tx)} dy \\
= \;& \epsilon^m \left(\frac{1}{i}\right)^{\frac{mn}{2}} \big(\det\mathcal{M}\big)^{\frac{n}{2}} \int_{\mathbb{R}^{(m,n)}} e^{\pi\, i\,\sigma\{\mathcal{M}(y\,\Omega\,{}^ty+2\,y\,{}^tZ)\}}\, e^{-2\pi\, i\,\sigma(\mathcal{M}\,y\,{}^tx)} dy \\
= \;& \epsilon^m \left(\frac{1}{i}\right)^{\frac{mn}{2}} \big(\det\mathcal{M}\big)^{\frac{n}{2}} \int_{\mathbb{R}^{(m,n)}} e^{\pi\, i\,\sigma\big\{\mathcal{M}\big(y\,\Omega\,{}^ty\,+\,2\,y\,{}^t(Z-x)\big)\big\}} dy.
\end{aligned}
$$

If we substitute $u=\mathcal{M}^{\frac{1}{2}}\,y$, then $du = (\det\mathcal{M})^{\frac{n}{2}}\;dy$. Therefore according to Lemma 5.1, we obtain

$$
\begin{aligned}
&\Big(\omega_{\mathcal{M}}\big(\widetilde{g}_*\big)\mathscr{F}^{(\mathcal{M})}_{\Omega,Z}\Big)(x)\\
=\ & \epsilon^m \left(\frac{1}{i}\right)^{\frac{mn}{2}} \big(\det\mathcal{M}\big)^{\frac{n}{2}} \int_{\mathbb{R}^{(m,n)}} e^{\pi\, i\,\sigma\left(u\,\Omega\,{}^tu+2\,\mathcal{M}^{1/2}\,u\,{}^t(Z-x)\right)}\,(\det\mathcal{M})^{-\frac{n}{2}}\,du\\
=\ & \epsilon^m \left(\frac{1}{i}\right)^{\frac{mn}{2}} \int_{\mathbb{R}^{(m,n)}} e^{\pi\, i\,\sigma\left(u\,\Omega\,{}^tu+2\,u\,{}^t(\mathcal{M}^{1/2}\,(Z-x))\right)}\,du\\
=\ & \epsilon^m \left(\frac{1}{i}\right)^{\frac{mn}{2}} \left(\det\frac{\Omega}{i}\right)^{-\frac{m}{2}} e^{-\pi\, i\,\sigma\left\{\mathcal{M}^{1/2}(Z-x)\,\Omega^{-1}\,{}^t(Z-x)\,\mathcal{M}^{1/2}\right\}}\quad(\text{by Lemma 5.1})\\
=\ & \epsilon^m\ (\det\Omega)^{-\frac{m}{2}}\ e^{-\pi\, i\,\sigma\left(\mathcal{M}\,(Z-x)\,\Omega^{-1}\,{}^t(Z-x)\right)}\\
=\ & \epsilon^m\ (\det\Omega)^{-\frac{m}{2}}\ e^{-\pi\, i\,\sigma\left(\mathcal{M}(Z\,\Omega^{-1}\,{}^tZ+x\,\Omega^{-1}\,{}^tx-2\,Z\,\Omega^{-1}\,{}^tx)\right)}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&J_{1/2}\big((\sigma_n,\epsilon),\Omega\big)\\
=\ & \epsilon^{-1}\det{}^{-1/2}\left(\frac{\sigma_n\cdot\Omega-\overline{\sigma_n\cdot(iI_n)}}{2\,i}\right)\det{}^{-1/2}\left(\frac{\Omega-\overline{(iI_n)}}{2\,i}\right)|J(\sigma_n,iI_n)|^{-1/2}\\
=\ & \epsilon^{-1}\det{}^{-1/2}\left(\frac{-i\,\Omega^{-1}\big(\Omega-\overline{i\,I_n}\big)}{2\,i}\right)\det{}^{-1/2}\left(\frac{\Omega-\overline{(iI_n)}}{2\,i}\right)(i^n)^{-1/2}\\
=\ & \epsilon^{-1}\det{}^{-1/2}\left(-i\,\Omega^{-1}\right)\,i^{-n/2}\\
=\ & \epsilon^{-1}\left(\frac{1}{i}\right)^{-n/2}(\det\,\Omega)^{1/2}\ i^{-n/2}\\
=\ & \epsilon^{-1}\,(\det\,\Omega)^{1/2}.
\end{aligned}
$$

Therefore, according to Formula (5.3), for $x\in\mathbb{R}^{(m,n)}$, we obtain

$$
\begin{aligned}
&J^*_{\mathcal{M}}\big(\widetilde{g}_*,(\Omega,Z)\big)^{-1}\mathscr{F}^{(\mathcal{M})}_{\widetilde{g}_*\cdot(\Omega,Z)}(x)\\
=\ & e^{-\pi\, i\,\sigma(\mathcal{M}\,Z\,\Omega^{-1}\,{}^tZ)}\ J_{1/2}\big((\sigma_n,\epsilon),\Omega\big)^{-m}\ \mathscr{F}^{(\mathcal{M})}_{-\Omega^{-1},\,Z\,\Omega^{-1}}(x)\\
=\ & \epsilon^m\ (\det\Omega)^{-\frac{m}{2}}\ e^{-\pi\, i\,\sigma(\mathcal{M}\,Z\,\Omega^{-1}\,{}^tZ)}\ e^{\pi\, i\,\sigma\left\{\mathcal{M}\left(x\,(-\Omega^{-1})\,{}^tx+2\,x\,{}^t(Z\,\Omega^{-1})\right)\right\}}\\
=\ & \epsilon^m\ (\det\Omega)^{-\frac{m}{2}}\ e^{-\pi\, i\,\sigma\left(\mathcal{M}(Z\,\Omega^{-1}\,{}^tZ+x\,\Omega^{-1}\,{}^tx-2\,Z\,\Omega^{-1}\,{}^tx)\right)}.
\end{aligned}
$$

Hence we prove the covariance relation (5.4) in the case $\widetilde{g}_*=\big((\sigma_n,\epsilon),(0,0;0)\big)$ with $\epsilon^2=(-i)^n$. Since $J^*_{\mathcal{M}}$ is an automorphic factor for G^J_* on $\mathbb{H}_{n,m}$, we see that if the covariance relation (5.4) holds for two elements $\widetilde{g}_*$, $\widetilde{h}_*$ in G^J_*, then it holds for $\widetilde{g}_*\widetilde{h}_*$. Finally we complete the proof. $\square$

It is natural to raise the following question :

Problem : Find all the covariant maps for the Schödinger-Weil representation $\omega_{\mathcal{M}}$ on $\mathbb{H}_{n,m}$ and the automorphic factor $J^*_{\mathcal{M}}$.

6. Construction of Jacobi Forms of Half Integral Weight

Let (π, V_π) be a unitary representation of G^J_* on the representation space V_π. Let Γ be an arithmetic subgroup of the Siegel modular group Γ_n. We set $\Gamma_* = \pi_*^{-1}(\Gamma)$ and

$$\Gamma^J_* = \Gamma_* \ltimes H_{\mathbb{Z}}^{0n,m)}.$$

We assume that (π, V_π) satisfies the following conditions (A) and (B):

(A) There exists a vector valued map

$$\mathscr{F} : \mathbb{H}_{n,m} \longrightarrow V_\pi, \quad (\Omega, Z) \mapsto \mathscr{F}_{\Omega,Z} := \mathscr{F}(\Omega, Z)$$

satisfying the following covariance relation

$$\pi(\widetilde{g}_*)\mathscr{F}_{\Omega,Z} = \psi(\widetilde{g}_*)\, J_*\big(\widetilde{g}_*, (\Omega, Z)\big)^{-1}\, \mathscr{F}_{\widetilde{g}_*\cdot(\Omega,Z)} \quad \text{for all } \widetilde{g}_* \in G^J_*,\ (\Omega, Z) \in \mathbb{H}_{n,m}, \tag{6.1}$$

where ψ is a character of G^J_* and $J_* : G^J_* \times \mathbb{H}_{n,m} \longrightarrow \mathbb{C}^\times$ is a certain automorphic factor for G^J_* on $\mathbb{H}_{n,m}$.

(B) There exists a linear functional $\theta : V_\pi \longrightarrow \mathbb{C}$ which is semi-invariant under the action of Γ^J_*, in other words, for all $\widetilde{\gamma}_* \in \Gamma^J_*$ and $(\Omega, Z) \in \mathbb{H}_{n,m}$,

$$\langle\, \pi^*(\widetilde{\gamma}_*)\theta,\, \mathscr{F}_{\Omega,Z}\, \rangle = \langle\, \theta, \pi(\widetilde{\gamma}_*)^{-1}\mathscr{F}_{\Omega,Z}\, \rangle = \chi(\widetilde{\gamma}_*)\, \langle\, \theta,\, \mathscr{F}_{\Omega,Z}\, \rangle, \tag{6.2}$$

where π^* is the contragredient of π and $\chi : \Gamma^J_* \longrightarrow T$ is a unitary character of Γ^J_*.

Under the assumptions (A) and (B) on a unitary representation (π, V_π), we define the function Θ on $\mathbb{H}_{n,m}$ by

$$\Theta(\Omega, Z) := \langle\, \theta, \mathscr{F}_{\Omega,Z}\, \rangle = \theta\big(\mathscr{F}_{\Omega,Z}\big), \quad (\Omega, Z) \in \mathbb{H}_{n,m}. \tag{6.3}$$

We now shall see that Θ is an automorphic form on $\mathbb{H}_{n,m}$ with respect to Γ^J_* for the automorphic factor J_*.

Lemma 6.1. *Let (π, V_π) be a unitary representation of G^J_* satisfying the above assumptions (A) and (B). Then the function Θ on $\mathbb{H}_{n,m}$ defined by (6.3) satisfies the following modular transformation behavior*

$$\Theta\big(\widetilde{\gamma}_* \cdot (\Omega, Z)\big) = \psi(\widetilde{\gamma}_*)^{-1}\, \chi(\widetilde{\gamma}_*)^{-1}\, J_*\big(\widetilde{\gamma}_*, (\Omega, Z)\big)\, \Theta(\Omega, Z) \tag{6.4}$$

for all $\widetilde{\gamma}_* \in \Gamma^J_*$ and $(\Omega, Z) \in \mathbb{H}_{n,m}$.

Proof. For any $\widetilde{\gamma}_* \in \Gamma^J_*$ and $(\Omega, Z) \in \mathbb{H}_{n,m}$, according to the assumptions (6.1) and (6.2), we obtain

$$\begin{aligned}
&\Theta\big(\widetilde{\gamma}_* \cdot (\Omega, Z)\big) = \big\langle\, \theta, \mathscr{F}_{\widetilde{\gamma}_*\cdot(\Omega,Z)} \big\rangle \\
=\ & \big\langle\, \theta, \psi(\widetilde{\gamma}_*)^{-1} J_*\big(\widetilde{\gamma}_*, (\Omega, Z)\big)\, \pi(\widetilde{\gamma}_*)\mathscr{F}_{\Omega,Z}\, \big\rangle \\
=\ & \psi(\widetilde{\gamma}_*)^{-1} J_*\big(\widetilde{\gamma}_*, (\Omega, Z)\big)\, \big\langle\, \theta, \pi(\widetilde{\gamma}_*)\mathscr{F}_{\Omega,Z}\, \big\rangle \\
=\ & \psi(\widetilde{\gamma}_*)^{-1}\, \chi(\widetilde{\gamma}_*)^{-1}\, J_*\big(\widetilde{\gamma}_*, (\Omega, Z)\big)\, \big\langle\, \theta, \mathscr{F}_{\Omega,Z}\, \big\rangle \\
=\ & \psi(\widetilde{\gamma}_*)^{-1}\, \chi(\widetilde{\gamma}_*)^{-1}\, J_*\big(\widetilde{\gamma}_*, (\Omega, Z)\big)\, \Theta(\Omega, Z).
\end{aligned}$$

□

Now for a positive definite integral symmetric matrix $\mathcal{M}$ of degree m, we define the holomorphic function $\Theta_{\mathcal{M}} : \mathbb{H}_{n,m} \longrightarrow \mathbb{C}$ by

$$\Theta_{\mathcal{M}}(\Omega, Z) := \sum_{\xi \in \mathbb{Z}^{(m,n)}} e^{\pi\, i\, \sigma\left(\mathcal{M}(\xi\,\Omega\,{}^t\xi\, +\, 2\,\xi\,{}^tZ)\right)}, \quad (\Omega, Z) \in \mathbb{H}_{n,m}. \tag{6.5}$$

Theorem 6.1. *Let m be an odd positive integer. Let $\mathcal{M}$ be a symmetric positive definite integral matrix of degree m such that* $\det \mathcal{M} = 1$. *Let Γ be an arithmetic subgroup of Γ_n generated by all the following elements*

$$t(b) = \begin{pmatrix} I_n & b \\ 0 & I_n \end{pmatrix}, \quad g(\alpha) = \begin{pmatrix} {}^t\alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \quad \sigma_n = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

where $b = {}^tb \in S(n, \mathbb{Z})$ with even diagonal and $\alpha \in GL(n, \mathbb{Z})$. Then for any $\widetilde{\gamma}_* \in \Gamma^J_*$, the function $\Theta_{\mathcal{M}}$ satisfies the functional equation

$$\Theta_{\mathcal{M}}\big(\widetilde{\gamma}_*\cdot(\Omega, Z)\big) = \rho_{\mathcal{M}}(\widetilde{\gamma}_*)\, J^*_{\mathcal{M}}\big(\widetilde{\gamma}_*, (\Omega, Z)\big)\Theta_{\mathcal{M}}(\Omega, Z), \quad (\Omega, Z) \in \mathbb{H}_{n,m}, \tag{6.6}$$

where $\rho_{\mathcal{M}}$ is a character of Γ^J_* and $J^*_{\mathcal{M}} : G^J_* \times \mathbb{H}_{n,m} \longrightarrow \mathbb{C}^\times$ is the automorphic factor for G^J_* on $\mathbb{H}_{n,m}$ defined by the formula (5.3).

Proof. For an element $\widetilde{\gamma}_* = \big((\gamma, \epsilon), (\lambda, \mu; \kappa)\big) \in \Gamma^J_*$ with $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$ and $(\lambda, \mu; \kappa) \in H^{(n,m)}_{\mathbb{Z}}$, we put $(\Omega_*, Z_*) = \widetilde{\gamma}_*\cdot(\Omega, Z)$ for $(\Omega, Z) \in \mathbb{H}_{n,m}$. Then we have

$$\begin{aligned} \Omega_* &= \gamma \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1}, \\ Z_* &= (Z + \lambda\,\Omega + \mu)(C\Omega + D)^{-1}. \end{aligned}$$

We define the linear functional ϑ on $L^2\big(\mathbb{R}^{(m,n)}\big)$ by

$$\vartheta(f) = \langle \vartheta, f\rangle := \sum_{\xi \in \mathbb{Z}^{(m,n)}} f(\xi), \qquad f \in L^2\big(\mathbb{R}^{(m,n)}\big).$$

We note that $\Theta_{\mathcal{M}}(\Omega, Z) = \vartheta\big(\mathscr{F}^{(\mathcal{M})}_{\Omega,Z}\big)$. Since $\mathscr{F}^{(\mathcal{M})}$ is a covariant map for the Schrödinger-Weil representation $\omega_{\mathcal{M}}$ by Theorem 5.1, according to Lemma 6.1, it suffices to prove that ϑ is semi-invariant for $\omega_{\mathcal{M}}$ under the action of Γ^J_*, in other words, ϑ satisfies the following semi-invariance relation

$$\Big\langle \vartheta, \omega_{\mathcal{M}}\big(\widetilde{\gamma}_*\big)\mathscr{F}^{(\mathcal{M})}_{\Omega,Z} \Big\rangle = \rho_{\mathcal{M}}\big(\widetilde{\gamma}_*\big)^{-1} \Big\langle \vartheta, \mathscr{F}^{(\mathcal{M})}_{\Omega,Z} \Big\rangle \tag{6.7}$$

for all $\widetilde{\gamma}_* \in \Gamma^J_*$ and $(\Omega, Z) \in \mathbb{H}_{n,m}$.

We note that the following elements $h(\lambda, \mu; \kappa) - \epsilon,\ t(b; \epsilon),\ g(\alpha; \epsilon)$ and $\sigma_{n,\epsilon}$ of Γ^J_* defined by

$$\begin{aligned} &h(\lambda, \mu; \kappa)_\epsilon = \big((I_{2n}, \epsilon), (\lambda, \mu; \kappa)\big) \text{ with } \epsilon = \pm 1,\ \lambda, \mu \in \mathbb{Z}^{(m,n)} \text{ and } \kappa \in \mathbb{Z}^{(m,m)}, \\ &t(b; \epsilon) = \big((t(b), \epsilon), (0, 0; 0)\big) \text{ with } b = {}^tb \in S(n, \mathbb{Z}) \text{ even diagonal}, \\ &g(\alpha; \epsilon) = \big((g(\alpha), \epsilon), (0, 0; 0)\big) \text{ with } \epsilon = \pm 1,\ \pm i,\ \alpha \in GL(n, \mathbb{Z}), \\ &\sigma_{n,\epsilon} = \big((\sigma_n, \epsilon), (0, 0; 0)\big) \text{ with } \epsilon^2 = (-i)^n. \end{aligned}$$

generate the Jacobi modular group Γ^J_*. Therefore it suffices to prove the semi-invariance relation (6.7) for the above generators of Γ^J_*.

Case I. $\widetilde{\gamma}_* = h(\lambda,\mu;\kappa)_\epsilon$ with $\epsilon = \pm 1,\ \lambda,\mu \in \mathbb{Z}^{(m,n)},\ \kappa \in \mathbb{Z}^{(m,m)}$.

In this case, we have

$$\Omega_* = \Omega \quad \text{and} \quad Z_* = Z + \lambda\,\Omega + \mu.$$

We put

$$h(\lambda,\mu;\kappa)_+ := \big((I_{2n},1),(\lambda,\mu;\kappa)\big)$$

and

$$h(\lambda,\mu;\kappa)_- := \big((I_{2n},-1),(\lambda,\mu;\kappa)\big).$$

It is easily seen that according to Formula (4.15), we obtain

$$J_{1/2}((I_{2n},1),\Omega) = 1 \quad \text{and} \quad J_{1/2}((I_{2n},-1),\Omega) = -1.$$

Therefore we get

$$J^*_{\mathcal{M}}\big(h(\lambda,\mu;\kappa)_+,(\Omega,Z)\big) = e^{-\pi i\,\sigma\{\mathcal{M}(\lambda\,\Omega\,{}^t\lambda + 2\,\lambda\,{}^tZ + \kappa + \mu\,{}^t\lambda)\}}$$

and

$$J^*_{\mathcal{M}}\big(h(\lambda,\mu;\kappa)_-,(\Omega,Z)\big) = -e^{-\pi i\,\sigma\{\mathcal{M}(\lambda\,\Omega\,{}^t\lambda + 2\,\lambda\,{}^tZ + \kappa + \mu\,{}^t\lambda)\}}.$$

According to the covariance relation (5.4),

$$\begin{aligned}
& \big\langle\, \vartheta, \omega_{\mathcal{M}}\big(h(\lambda,\mu;\kappa)_+\big)\mathscr{F}^{(\mathcal{M})}_{\Omega,Z} \,\big\rangle \\
= \ & \big\langle\, \vartheta, J^*_{\mathcal{M}}\big(h(\lambda,\mu;\kappa)_+,(\Omega,Z)\big)^{-1}\mathscr{F}^{(\mathcal{M})}_{h(\lambda,\mu;\kappa)_+\cdot(\Omega,Z)} \,\big\rangle \\
= \ & J^*_{\mathcal{M}}\big(h(\lambda,\mu;\kappa)_+,(\Omega,Z)\big)^{-1} \big\langle\, \vartheta, \mathscr{F}^{(\mathcal{M})}_{\Omega,Z+\lambda\,\Omega+\mu} \,\big\rangle \\
= \ & J^*_{\mathcal{M}}\big(h(\lambda,\mu;\kappa)_+,(\Omega,Z)\big)^{-1} \sum_{A\in\mathbb{Z}^{(m,n)}} e^{\pi i\,\sigma\left\{\mathcal{M}\left(A\Omega\,{}^tA + 2\,A\,{}^t(Z+\lambda\,\Omega+\mu)\right)\right\}} \\
= \ & J^*_{\mathcal{M}}\big(h(\lambda,\mu;\kappa)_+,(\Omega,Z)\big)^{-1} \cdot e^{-\pi i\,\sigma\left(\mathcal{M}(\lambda\,\Omega\,{}^t\lambda + 2\,\lambda\,{}^tZ)\right)} \\
& \times \sum_{A\in\mathbb{Z}^{(m,n)}} e^{2\pi i\,\sigma(\mathcal{M}A\,{}^t\mu)} e^{\pi i\,\sigma\left\{\mathcal{M}\left((A+\lambda)\,\Omega\,{}^t(A+\lambda) + 2\,(A+\lambda)\,{}^tZ\right)\right\}} \\
= \ & e^{\pi i\,\sigma\left(\mathcal{M}(\kappa + \mu\,{}^t\lambda)\right)} \big\langle\, \vartheta, \mathscr{F}^{(\mathcal{M})}_{\Omega,Z} \,\big\rangle.
\end{aligned}$$

Here we used the fact that $\sigma(\mathcal{M}A\,{}^t\mu)$ is an integer. In a similar way, we get

$$\big\langle\, \vartheta, \omega_{\mathcal{M}}\big(h(\lambda,\mu;\kappa)_-\big)\mathscr{F}^{(\mathcal{M})}_{\Omega,Z} \,\big\rangle = -e^{\pi i\,\sigma\left(\mathcal{M}(\kappa + \mu\,{}^t\lambda)\right)} \big\langle\, \vartheta, \mathscr{F}^{(\mathcal{M})}_{\Omega,Z} \,\big\rangle.$$

We put

$$\rho_{\mathcal{M}}\big(h(\lambda,\mu;\kappa)_+\big) = e^{-\pi i\,\sigma\left(\mathcal{M}(\kappa + \mu\,{}^t\lambda)\right)}$$

and

$$\rho_{\mathcal{M}}\big(h(\lambda,\mu;\kappa)_-\big) = -e^{-\pi i\,\sigma\left(\mathcal{M}(\kappa + \mu\,{}^t\lambda)\right)}.$$

Therefore ϑ satisfies the semi-invariance relation (6.7) in the case $\widetilde{\gamma}_* = h(\lambda,\mu;\kappa)_\epsilon$ with $\epsilon = \pm 1,\ \lambda,\mu \in \mathbb{Z}^{(m,n)}$ and $\kappa \in \mathbb{Z}^{(m,m)}$.

Case II. $\widetilde{\gamma}_* = t(b;\epsilon)$ with $\epsilon = \pm 1,\ b = {}^tb \in S(n,\mathbb{Z})$ even diagonal.

In this case, we have

$$\Omega_* = \Omega + b \quad \text{and} \quad Z_* = Z.$$

We put

$$t(b)_+ = \big((t(b),1),(0,0;0)\big)$$

and

$$t(b)_- = \big((t(b),-1),(0,0;0)\big).$$

It is easily seen that

$$J_{1/2}((t(b),1),\Omega)=1 \quad \text{and} \quad J_{1/2}((t(b),-1),\Omega)=-1.$$

Therefore we get

$$J^*_{\mathcal{M}}\big(t(b)_+,(\Omega,Z)\big)=1 \quad \text{and} \quad J^*_{\mathcal{M}}\big(t(b)_-,(\Omega,Z)\big)=-1.$$

According to the covariance relation (5.4), we obtain

$$\begin{aligned}
&\big\langle\, \vartheta, \omega_{\mathcal{M}}\big(t(b)_+\big)\mathscr{F}^{(\mathcal{M})}_{\Omega,Z}\,\big\rangle \\
=\ &\big\langle\, \vartheta, J^*_{\mathcal{M}}\big(t(b)_+,(\Omega,Z)\big)^{-1}\mathscr{F}^{(\mathcal{M})}_{t(b)_+\cdot(\Omega,Z)}\,\big\rangle \\
=\ &\big\langle\, \vartheta, \mathscr{F}^{(\mathcal{M})}_{\Omega+b,Z}\,\big\rangle \\
=\ &\sum_{A\in\mathbb{Z}^{(m,n)}} e^{\pi\, i\, \sigma\left\{\mathcal{M}\left(A\,(\Omega+b)\,{}^tA\,+\,2\,A\,{}^tZ\right)\right\}} \\
=\ &\sum_{A\in\mathbb{Z}^{(m,n)}} e^{\pi\, i\, \sigma\left(\mathcal{M}\left(A\,\Omega\,{}^tA\,+\,2\,A\,{}^tZ\right)\right)}\cdot e^{\pi\, i\, \sigma(\mathcal{M}A\,b\,{}^tA)} \\
=\ &\sum_{A\in\mathbb{Z}^{(m,n)}} e^{\pi\, i\, \sigma\left(\mathcal{M}(A\,\Omega\,{}^tA\,+\,2\,A\,{}^tZ)\right)} \\
=\ &\big\langle\, \vartheta, \mathscr{F}^{(\mathcal{M})}_{\Omega,Z}\,\big\rangle.
\end{aligned}$$

Here we used the fact that $\sigma(\mathcal{M}A\,b\,{}^tA)$ is an even integer because b is even integral. In a similar way, we obtain

$$\big\langle\, \vartheta, \omega_{\mathcal{M}}\big(t(b)_-\big)\mathscr{F}^{(\mathcal{M})}_{\Omega,Z}\,\big\rangle = -\big\langle\, \vartheta, \mathscr{F}^{(\mathcal{M})}_{\Omega,Z}\,\big\rangle.$$

We put

$$\rho_{\mathcal{M}}\big(t(b)_+\big)=1 \quad \text{and} \quad \rho_{\mathcal{M}}\big(t(b)_-\big)=-1.$$

Therefore ϑ satisfies the semi-invariance relation (6.7) in the case $\widetilde{\gamma}_* = t(b;\epsilon)$ with $\epsilon = \pm 1,\ b\in S(n,\mathbb{Z})$.

Case III. $\widetilde{g}_* = \big((g(\alpha),\epsilon),(0,0;0)\big)$ with $\epsilon=\pm 1$ ($\det\alpha>0$) and $\epsilon=\pm\, i$ ($\det\alpha<0$). Here $\alpha\in GL(n,\mathbb{Z})$.

In this case, we have

$$\Omega_* = {}^t\alpha\,\Omega\,\alpha \quad \text{and} \quad Z_* = Z\alpha.$$

We put

$$\begin{aligned}
g(\alpha)_+ :\ &=\ \big((g(\alpha),1),(0,0;0)\big),\\
g(\alpha)_- :\ &=\ \big((g(\alpha),-1),(0,0;0)\big),\\
g(\alpha)^+ :\ &=\ \big((g(\alpha),i),(0,0;0)\big),\\
g(\alpha)^- :\ &=\ \big((g(\alpha),-i),(0,0;0)\big).
\end{aligned}$$

And we can show easily that

$$\begin{aligned} J_{1/2}\big((g(\alpha),1),\Omega\big) &= (\det\,\alpha)^{-1/2}, \\ J_{1/2}\big((g(\alpha),-1),\Omega\big) &= -(\det\,\alpha)^{-1/2}, \\ J_{1/2}\big((g(\alpha),i),\Omega\big) &= i\,(\det\,\alpha)^{-1/2}, \\ J_{1/2}\big((g(\alpha),-i),\Omega\big) &= -i\,(\det\,\alpha)^{-1/2}. \end{aligned}$$

According to the covariance relation (5.4), we obtain

$$\begin{aligned} &\big\langle\,\vartheta,\omega_{\mathcal{M}}\big(g(\alpha)_+\big)\mathscr{F}^{(\mathcal{M})}_{\Omega,Z}\,\big\rangle \\ =\;& \big\langle\,\vartheta, J^*_{\mathcal{M}}\big(g(\alpha)_+,(\Omega,Z)\big)^{-1}\mathscr{F}^{(\mathcal{M})}_{g(\alpha)_+\cdot(\Omega,Z)}\,\big\rangle \\ =\;& (\det\alpha)^{\frac{m}{2}}\,\big\langle\,\vartheta,\mathscr{F}^{(\mathcal{M})}_{{}^t\alpha\,\Omega\,\alpha,Z\,\alpha}\big\rangle \\ =\;& (\det\alpha)^{\frac{m}{2}}\sum_{A\in\mathbb{Z}^{(m,n)}}\mathscr{F}^{(\mathcal{M})}_{{}^t\alpha\,\Omega\,\alpha,Z\,\alpha}(A) \\ =\;& (\det\alpha)^{\frac{m}{2}}\sum_{A\in\mathbb{Z}^{(m,n)}} e^{\pi i\,\sigma\{\mathcal{M}(A\,{}^t\alpha\,\Omega\,{}^t(A\,{}^t\alpha)+2\,A\,{}^t\alpha\,{}^tZ)\}} \\ =\;& (\det\alpha)^{\frac{m}{2}}\,\big\langle\,\vartheta,\mathscr{F}^{(\mathcal{M})}_{\Omega,Z}\,\big\rangle. \end{aligned}$$

In a similar way we get

$$\begin{aligned} \big\langle\,\vartheta,\omega_{\mathcal{M}}\big(g(\alpha)_-\big)\mathscr{F}^{(\mathcal{M})}_{\Omega,Z}\,\big\rangle &= (-1)^m\,(\det\alpha)^{\frac{m}{2}}\,\big\langle\,\vartheta,\mathscr{F}^{(\mathcal{M})}_{\Omega,Z}\,\big\rangle, \\ \big\langle\,\vartheta,\omega_{\mathcal{M}}\big(g(\alpha)^+\big)\mathscr{F}^{(\mathcal{M})}_{\Omega,Z}\,\big\rangle &= i^m\,(\det\alpha)^{\frac{m}{2}}\,\big\langle\,\vartheta,\mathscr{F}^{(\mathcal{M})}_{\Omega,Z}\,\big\rangle, \\ \big\langle\,\vartheta,\omega_{\mathcal{M}}\big(g(\alpha)^-\big)\mathscr{F}^{(\mathcal{M})}_{\Omega,Z}\,\big\rangle &= (-i)^m\,(\det\alpha)^{\frac{m}{2}}\,\big\langle\,\vartheta,\mathscr{F}^{(\mathcal{M})}_{\Omega,Z}\,\big\rangle. \end{aligned}$$

Now we put

$$\begin{aligned} \rho_{\mathcal{M}}\big(g(\alpha)_+\big) &= (\det\,\alpha)^{\frac{m}{2}}, \\ \rho_{\mathcal{M}}\big(g(\alpha)_-\big) &= (-1)^m(\det\,\alpha)^{\frac{m}{2}}, \\ \rho_{\mathcal{M}}\big(g(\alpha)^+\big) &= i^m\,(\det\,\alpha)^{\frac{m}{2}}, \\ \rho_{\mathcal{M}}\big(g(\alpha)^-\big) &= (-i)^m\,(\det\,\alpha)^{\frac{m}{2}}. \end{aligned}$$

Therefore ϑ satisfies the semi-invariance relation (6.7) in the case $\widetilde{\gamma}_* = g(\alpha;\epsilon)$ with $\epsilon = \pm 1,\ \pm i,\ \alpha\in GL(n,\mathbb{Z})$.

Case IV. $\widetilde{\gamma}_* = \big((\sigma_n,\epsilon),(0,0;0)\big)$ with $\sigma_n = \begin{pmatrix} 0 & -I_n \\ I_n & 0\end{pmatrix}$ and $\epsilon^2 = (-i)^n$.

In this case, we have

$$\Omega_* = -\Omega^{-1} \quad\text{and}\quad Z_* = Z\,\Omega^{-1}.$$

In the process of the proof of Theorem 5.1, using Lemma 5.1, we already showed that

$$\int_{\mathbb{R}^{(m,n)}} e^{\pi\,i\,\sigma(\mathcal{M}(y\,\Omega\,{}^ty+2\,y\,{}^tZ))}dy = \big(\det\mathcal{M}\big)^{-\frac{n}{2}}\left(\det\frac{\Omega}{i}\right)^{-\frac{m}{2}} e^{-\pi\,i\,\sigma(\mathcal{M}\,Z\,\Omega^{-1}\,{}^tZ)}. \tag{6.8}$$

By (6.8), we see that the Fourier transform of $\mathscr{F}^{(\mathcal{M})}_{\Omega,Z}$ is given by

$$\widehat{\mathscr{F}^{(\mathcal{M})}_{\Omega,Z}}(x) = \big(\det \mathcal{M}\big)^{-\frac{n}{2}} \left(\det \frac{\Omega}{i}\right)^{-\frac{m}{2}} e^{-\pi\, i\, \sigma(\mathcal{M}\,(Z-x)\,\Omega^{-1}\,{}^t(Z-x))}, \tag{6.9}$$

where $\widehat{f}$ is the Fourier transform of f defined by

$$\widehat{f}(x) = \int_{\mathbb{R}}^{(m,n)} f(y)\, e^{-2\,\pi\, i\, \sigma(y\,{}^t x)}\, dy, \quad x \in \mathbb{R}^{(m,n)}.$$

On the other hand, in the process of the proof of Case IV in Theorem 5.1, we showed that

$$J^*_{\mathcal{M}}\big(\widetilde{g}_*, (\Omega, Z)\big) = \epsilon^{-m}\, (\det\, \Omega)^{\frac{m}{2}}\, e^{\pi\, i\, \sigma(\mathcal{M} Z\, \Omega^{-1\,t} Z)}.$$

According to the covariance relation (5.4), Formula (6.9) and Poisson summation formula, we obtain

$$\begin{aligned}
& \big\langle\, \vartheta, \omega_{\mathcal{M}}\big(\widetilde{\gamma}_*\big) \mathscr{F}^{(\mathcal{M})}_{\Omega,Z}\, \big\rangle \\
=\ & \big\langle\, \vartheta, J^*_{\mathcal{M}}\big(\widetilde{\gamma}_*, (\Omega, Z)\big)^{-1} \mathscr{F}^{(\mathcal{M})}_{\widetilde{\gamma}_*\cdot(\Omega,Z)}\, \big\rangle \\
=\ & J^*_{\mathcal{M}}\big(\widetilde{\gamma}_*, (\Omega, Z)\big)^{-1} \big\langle\, \vartheta, \mathscr{F}^{(\mathcal{M})}_{-\Omega^{-1}, Z\Omega^{-1}}\, \big\rangle \\
=\ & \epsilon^m\, (\det \Omega)^{-\frac{m}{2}}\, e^{-\pi\, i\, \sigma(\mathcal{M}\, Z\, \Omega^{-1\,t} Z)} \sum_{A \in \mathbb{Z}^{(m,n)}} e^{-\pi\, i\, \sigma\left(\mathcal{M}(A\,\Omega^{-1\,t}A \,-\, 2\,A\,\Omega^{-1\,t}Z)\right)} \\
=\ & \epsilon^m\, (\det \Omega)^{-\frac{m}{2}} \sum_{A \in \mathbb{Z}^{(m,n)}} e^{-\pi\, i\, \sigma\left(\mathcal{M}(Z\,\Omega^{-1\,t}Z \,+\, A\,\Omega^{-1\,t}A \,-\, 2\,A\,\Omega^{-1\,t}Z)\right)} \\
=\ & \epsilon^m\, (\det \Omega)^{-\frac{m}{2}} \sum_{A \in \mathbb{Z}^{(m,n)}} e^{-\pi\, i\, \sigma\left(\mathcal{M}(Z-A)\,\Omega^{-1\,t}(Z-A)\right)} \\
=\ & \epsilon^m\, (\det \Omega)^{-\frac{m}{2}} \big(\det \mathcal{M}\big)^{\frac{n}{2}} \left(\det \frac{\Omega}{i}\right)^{\frac{m}{2}} \sum_{A \in \mathbb{Z}^{(m,n)}} \widehat{\mathscr{F}^{(\mathcal{M})}_{\Omega,Z}}(\mathcal{M}\, A) \quad (\text{by Formula } (6.9)) \\
=\ & \epsilon^m \left(\det \frac{I_n}{i}\right)^{\frac{m}{2}} \sum_{A \in \mathbb{Z}^{(m,n)}} \widehat{\mathscr{F}^{(\mathcal{M})}_{\Omega,Z}}(A) \quad (\text{because } \det \mathcal{M} = 1) \\
=\ & \epsilon^m \left(\det \frac{I_n}{i}\right)^{\frac{m}{2}} \sum_{A \in \mathbb{Z}^{(m,n)}} \mathscr{F}^{(\mathcal{M})}_{\Omega,Z}(A) \quad (\text{by Poisson summation formula}) \\
=\ & \epsilon^m\, (-i)^{\frac{mn}{2}} \big\langle\, \vartheta, \mathscr{F}^{(\mathcal{M})}_{\Omega,Z}\, \big\rangle.
\end{aligned}$$

We put

$$\rho_{\mathcal{M}}(\widetilde{\gamma}_*) = \epsilon^m\, (-i)^{\frac{mn}{2}}.$$

Therefore ϑ satisfies the semi-invariance relation (6.7) in the case $\widetilde{\gamma}_* = \big((\sigma_n, \epsilon), (0, 0; 0)\big)$ with $\epsilon^2 = (-i)^n$. The proof of Case IV is completed.

Since $J^*_{\mathcal{M}}$ is an automorphic factor for G^J_* on $\mathbb{H}_{n,m}$, we see that if the formula (6.6) holds for two elements $\widetilde{\gamma}_1, \widetilde{\gamma}_2$ in Γ^J, then it holds for $\widetilde{\gamma}_1\widetilde{\gamma}_2$. Finally we complete the proof of Theorem 6.1. $\square$

Corollary 6.1. *Let Γ_*^J and $\rho_{\mathcal M}$ be as before in Theorem 6.1. If m is odd, then $\Theta_{\mathcal M}(\Omega, Z)$ is a Jacobi form of a half integral weight $\frac{m}{2}$ and index $\frac{\mathcal M}{2}$ with respect to an arithmetic subgroup Γ_*^J for a character $\rho_{\mathcal M}$ of Γ_*^J.*

Remark 6.1. *Let $a = (a_1, a_2) \in \mathbb{Z}^n \times \mathbb{Z}^n$ with $a_1, a_2 \in \mathbb{Z}^n$. Takase* [21] *considered the following theta series defined by*

$$\vartheta_a(\Omega, Z) := \sum_{\ell \in \mathbb{Z}^n} e^{\pi\, i\, ((\ell + a_1)\, \Omega\, {}^t(\ell + a_2) + 2\, (\ell + a_1)\, {}^t(Z + a_2))},$$

where $\Omega \in \mathbb{H}_n$ and $Z \in \mathbb{C}^n$. We put

$$\vartheta_a^*(\Omega, Z) := e^{-a_1\, {}^t a_2}\, \vartheta_a(\Omega, Z).$$

We let Γ_0 be an arithmetic subgroup of Γ_n consisting of $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ such that

(1) $L\gamma = L$, where $L = \mathbb{Z}^n \times \mathbb{Z}^n$.

(2) $(xA + y\,C)\, {}^t(xB + yD) \equiv x\, {}^t y \pmod{2\mathbb{Z}}$ for all $(x, y) \in L$.

We put

$$\Gamma_{*,0} = \pi_*^{-1}(\Gamma_0) \quad \text{and} \quad \Gamma_{*,0}^J = \Gamma_{*,0} \ltimes H_{\mathbb{Z}}^{(n,1)}.$$

He proved that for any $\widetilde{\gamma}_* = \big((\gamma, \epsilon), (\lambda, \mu; t)\big) \in \Gamma_{*,0}^J$ with $\gamma \in \Gamma_0$, the following transformation formula

$$\vartheta_{a\gamma^{-1}}^*\big(\widetilde{\gamma}_* \cdot (\Omega, Z)\big) = \rho((\gamma, \epsilon))\, \chi_a((\lambda, \mu; t))\, J_{\mathbf 1}(\widetilde{\gamma}_*, (\Omega, Z))\, \vartheta_a^*(\Omega, Z)$$

holds, where χ_a denotes the unitary character of $L \times \mathbb{R}$ defined by

$$\chi_a((\lambda, \mu; t)) = e^{2\,\pi\, i\, \left(t + \frac{1}{2}\, \lambda\, {}^t\mu - \lambda\, {}^t a_2 + a_1\, {}^t\mu\right)}, \quad (\lambda, \mu; t) \in L \times \mathbb{R}$$

and $\mathbf{1} = (1)$ denotes the 1×1 matrix. Here $\rho : \Gamma_{*,0} \longrightarrow T$ is a certain unitary character that is given explicitly in [21, Theorem 5.3, p.134].

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