THE SCHRÖDINGER-WEIL REPRESENTATION AND THETA SUMS

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Abstract. In this paper, we construct the Schrödinger-Weil representation of the Jacobi group associated with a positive definite symmetric real matrix of degree m and as its application, we obtain some properties of theta sums associated with the Schrödinger-Weil representation.

1. Introduction

For a given fixed positive integer n, we let

$$\mathbb{H}_n = \left\{ \Omega \in \mathbb{C}^{(n,n)} \,\middle|\, \Omega = {}^t\Omega, \quad \mathrm{Im}\,\Omega > 0 \right\}$$

be the Siegel upper half plane of degree n and let

$$Sp(n,\mathbb{R}) = \left\{ g \in \mathbb{R}^{(2n,2n)} \,\middle|\, {}^tgJ_ng = J_n \right\}$$

be the symplectic group of degree n, where $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F for two positive integers k and l, ${}^tM$ denotes the transpose of a matrix M, $\mathrm{Im}\,\Omega$ denotes the imaginary part of Ω and

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Here I_n denotes the identity matrix of degree n. We see that $Sp(n,\mathbb{R})$ acts on $\mathbb{H}_n$ transitively by

$$g \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1},$$

where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R})$ and $\Omega \in \mathbb{H}_n$.

For two positive integers n and m, we consider the Heisenberg group

$$H_{\mathbb{R}}^{(n,m)} = \{ (\lambda,\mu;\kappa) \mid \lambda,\mu \in \mathbb{R}^{(m,n)}, \ \kappa \in \mathbb{R}^{(m,m)}, \ \ \kappa + \mu\, {}^t\lambda \text{ symmetric } \}$$

endowed with the following multiplication law

$$(\lambda,\mu;\kappa) \circ (\lambda',\mu';\kappa') = (\lambda+\lambda', \mu+\mu'; \kappa+\kappa' + \lambda\, {}^t\mu' - \mu\, {}^t\lambda').$$

We let

$$G^J = Sp(n,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)} \qquad \text{(semi-direct product)}$$

be the Jacobi group endowed with the following multiplication law

$$\Big(g,(\lambda,\mu;\kappa)\Big) \cdot \Big(g',(\lambda',\mu';\kappa')\Big) = \Big(gg',(\widetilde{\lambda}+\lambda', \widetilde{\mu}+\mu'; \kappa+\kappa' + \widetilde{\lambda}\, {}^t\mu' - \widetilde{\mu}\, {}^t\lambda')\Big)$$

2010 Mathematics Subject Classification: Primary 11F27, 11F50

Keywords and phrases: the Schrödinger representation, the Schrödinger-Weil representation, theta sums.

The author was supported by Basic Science Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (47724-1).

with $g,g'\in Sp(n,\mathbb{R}),(\lambda,\mu;\kappa),\ (\lambda',\mu';\kappa')\in H_{\mathbb{R}}^{(n,m)}$ and $(\widetilde{\lambda},\widetilde{\mu})=(\lambda,\mu)g'$. Then we have the *natural transitive action* of G^J on the Siegel-Jacobi space $\mathbb{H}_{n,m}:=\mathbb{H}_n\times\mathbb{C}^{(m,n)}$ defined by

$$\Big(g,(\lambda,\mu;\kappa)\Big)\cdot(\Omega,Z)=\Big((A\Omega+B)(C\Omega+D)^{-1},(Z+\lambda\,\Omega+\mu)(C\,\Omega+D)^{-1}\Big),$$

where $g=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in Sp(n,\mathbb{R}),\ \ (\lambda,\mu;\kappa)\in H_{\mathbb{R}}^{(n,m)}$ and $(\Omega,Z)\in\mathbb{H}_{n,m}$. Thus $\mathbb{H}_{n,m}$ is a homogeneous Kähler space which is not symmetric. In fact, $\mathbb{H}_{n,m}$ is biholomorphic to the homogeneous space G^J/K^J, where $K^J\cong U(n)\times S(m,\mathbb{R})$. Here $U(n)$ denotes the unitary group of degree n and $S(m,\mathbb{R})$ denote the abelian additive group consisting of all $m\times m$ symmetric real matrices. We refer to [**?**, **?**, **?**], [**?**]-[**?**] for more details on materials related to the Siegel-Jacobi space, e.g., Jacobi forms, invariant metrics, invariant differential operators and Maass-Jacobi forms.

The Weil representation for a symplectic group was first introduced by A. Weil in [**?**] to reformulate Siegel's analytic theory of quadratic forms (cf. [**?**]) in terms of the group theoretical theory. It is well known that the Weil representation plays a central role in the study of the transformation behaviors of theta series. In this paper, we construct the Schrödinger-Weil representation $\omega_{\mathcal{M}}$ of the Jacobi group G^J associated with a positive definite symmetric real matrix $\mathcal{M}$ of degree n.

This paper is organized as follows. In Section 2, we review the Schrödinger representation of the Heisenberg group $H_{\mathbb{R}}^{(n,m)}$ associated with a nonzero symmetric real matrix of degree m which is formulated in [**?**, **?**, **?**]. In Section 3, we define the Schrödinger-Weil representation $\omega_{\mathcal{M}}$ of the Jacobi group G^J associated with a symmetric positive definite matrix $\mathcal{M}$ and provide some of the actions of $\omega_{\mathcal{M}}$ on the representation space $L^2\big(\mathbb{R}^{(m,n)}\big)$ explicitly. In the final section, we define the theta sum $\Theta_f^{[\mathcal{M}]}(\tau,\phi\,;\lambda,\mu,\kappa)$ and obtain some properties of the theta sum. The theta sum $\Theta_f^{[\mathcal{M}]}(\tau,\phi\,;\lambda,\mu,\kappa)$ is a generalization of the theta sum defined by J. Marklof [**?**].

Notations : We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of integers, the field of real numbers and the field of complex numbers respectively. $\mathbb{C}^\times$ denotes the multiplicative group of nonzero complex numbers and $\mathbb{Z}^\times$ denotes the set of all nonzero integers. T denotes the multiplicative group of complex numbers of modulus one. The symbol ":=" means that the expression on the right is the definition of that on the left. For two positive integers k and l, $F^{(k,l)}$ denotes the set of all $k\times l$ matrices with entries in a commutative ring F. For a square matrix $A\in F^{(k,k)}$ of degree k, $\sigma(A)$ denotes the trace of A. For any $M\in F^{(k,l)}$, ${}^tM$ denotes the transpose of a matrix M. I_n denotes the identity matrix of degree n. We put $i=\sqrt{-1}$. For a positive integer m we denote by $S(m,F)$ the additive group consisting of all $m\times m$ symmetric matrices with coefficients in a commutative ring F.

2. The Schrödinger Representation

First of all, we observe that $H_{\mathbb{R}}^{(n,m)}$ is a 2-step nilpotent Lie group. The inverse of an element $(\lambda,\mu;\kappa)\in H_{\mathbb{R}}^{(n,m)}$ is given by

$$(\lambda,\mu;\kappa)^{-1}=(-\lambda,-\mu;-\kappa+\lambda\,{}^t\mu-\mu\,{}^t\lambda).$$

Now we set

$$[\lambda,\mu;\kappa]=(0,\mu;\kappa)\circ(\lambda,0;0)=(\lambda,\mu;\kappa-\mu\,{}^t\lambda).$$

Then $H_{\mathbb{R}}^{(n,m)}$ may be regarded as a group equipped with the following multiplication

$$[\lambda,\mu;\kappa]\diamond[\lambda_0,\mu_0;\kappa_0]=[\lambda+\lambda_0,\mu+\mu_0;\kappa+\kappa_0+\lambda\,{}^t\mu_0+\mu_0\,{}^t\lambda].$$

The inverse of $[\lambda,\mu;\kappa]\in H_{\mathbb{R}}^{(n,m)}$ is given by

$$[\lambda,\mu;\kappa]^{-1}=[-\lambda,-\mu;-\kappa+\lambda\,{}^t\mu+\mu\,{}^t\lambda].$$

We set

$$L=\left\{\,[0,\mu;\kappa]\in H_{\mathbb{R}}^{(n,m)}\,\Big|\,\mu\in\mathbb{R}^{(m,n)},\ \kappa={}^t\kappa\in\mathbb{R}^{(m,m)}\,\right\}.$$

Then L is a commutative normal subgroup of $H_{\mathbb{R}}^{(n,m)}$. Let $\widehat{L}$ be the Pontrajagin dual of L, i.e., the commutative group consisting of all unitary characters of L. Then $\widehat{L}$ is isomorphic to the additive group $\mathbb{R}^{(m,n)}\times S(m,\mathbb{R})$ via the canonical pairing

$$\langle a,\hat{a}\rangle=e^{2\pi i\,\sigma(\hat{\mu}\,{}^t\mu+\hat{\kappa}\kappa)},\quad a=[0,\mu;\kappa]\in L,\ \hat{a}=(\hat{\mu},\hat{\kappa})\in\widehat{L},$$

where $S(m,\mathbb{R})$ denotes the space of all symmetric $m\times m$ real matrices.

We put

$$S=\left\{\,[\lambda,0;0]\in H_{\mathbb{R}}^{(n,m)}\,\Big|\,\lambda\in\mathbb{R}^{(m,n)}\,\right\}\cong\mathbb{R}^{(m,n)}.$$

Then S acts on L as follows:

$$[\lambda,0;0]*[0,\mu;\kappa]:=[0,\mu;\kappa+\lambda\,{}^t\mu+\mu\,{}^t\lambda],\qquad[\lambda,0,0]\in S,\ [0,\mu;\kappa]\in L.$$

We see that the Heisenberg group $\left(H_{\mathbb{R}}^{(n,m)},\diamond\right)$ is isomorphic to the semi-direct product $S\ltimes L$ of S and L whose multiplication law is defined by

$$\begin{aligned}&\big([\lambda,0;0],[0,\mu;\kappa]\big)\star\big([\lambda_0,0;0],[0,\mu_0;\kappa_0]\big)\\ :=\ &\big([\lambda+\lambda_0,0;0],[0,\mu+\mu_0;\kappa+\kappa_0+\lambda\,{}^t\mu_0+\mu_0\,{}^t\lambda]\big).\end{aligned}$$

On the other hand, S acts on $\widehat{L}$ by

$$[\lambda,0;0]\bullet(\hat{\mu},\hat{\kappa})=(\hat{\mu}+2\hat{\kappa}\lambda,\hat{\kappa}),$$

where $[\lambda,0;0]\in S,\ (\hat{\mu},\hat{\kappa})\in\widehat{L}$ with $\hat{\mu}\in\mathbb{R}^{(m,n)}$ and $\hat{\kappa}\in S(m,\mathbb{R})$. Then we have the following relation

$$\langle\,[\lambda,0;0]*[0,\mu;\kappa],(\hat{\mu},\hat{\kappa})\,\rangle=\langle\,[0,\mu;\kappa],[\lambda,0;0]\bullet(\hat{\mu},\hat{\kappa})\,\rangle,$$

where $[\lambda,0;0]\in S,\ [0,\mu;\kappa]\in L$ and $(\hat{\mu},\hat{\kappa})\in\widehat{L}$.

We have three types of S-orbits in $\widehat{L}$.

TYPE I. Let $\hat{\kappa}\in S(m,\mathbb{R})$ be nondegenerate. The S-orbit of $(0,\hat{\kappa})\in\widehat{L}$ is given by

$$\widehat{\mathcal{O}}_{\hat{\kappa}}=\left\{(2\hat{\kappa}\lambda,\hat{\kappa})\in\widehat{L}\,\Big|\,\lambda\in\mathbb{R}^{(m,n)}\right\}\cong\mathbb{R}^{(m,n)}.$$

TYPE II. Let $(\hat{\mu},\hat{\kappa})\in\mathbb{R}^{(m,n)}\times S(m,\mathbb{R})$ with $\hat{\mu}\in\mathbb{R}^{(m,n)},\ \hat{\kappa}\in S(m,\mathbb{R})$ and degenerate $\hat{\kappa}\neq 0$. Then

$$\widehat{\mathcal{O}}_{(\hat{\mu},\hat{\kappa})}=\left\{(\hat{\mu}+2\hat{\kappa}\lambda,\hat{\kappa})\,\Big|\,\lambda\in\mathbb{R}^{(m,n)}\right\}\subsetneqq\mathbb{R}^{(m,n)}\times\{\hat{\kappa}\}.$$

Type III. Let $\hat{y} \in \mathbb{R}^{(m,n)}$. The S-orbit $\widehat{\mathcal{O}}_{\hat{y}}$ of $(\hat{y}, 0)$ is given by

$$\widehat{\mathcal{O}}_{\hat{y}} = \{\, (\hat{y}, 0) \,\}\,.$$

We have

$$\widehat{L} = \left(\bigcup_{\substack{\hat{\kappa} \in S(m,\mathbb{R}) \\ \hat{\kappa} \ \text{nondegenerate}}} \widehat{\mathcal{O}}_{\hat{\kappa}} \right) \bigcup \left(\bigcup_{\hat{y} \in \mathbb{R}^{(m,n)}} \widehat{\mathcal{O}}_{\hat{y}} \right) \bigcup \left(\bigcup_{\substack{(\hat{\mu},\hat{\kappa}) \in \mathbb{R}^{(m,n)} \times S(m,\mathbb{R}) \\ \hat{\kappa} \neq 0 \ \text{degenerate}}} \widehat{\mathcal{O}}_{(\hat{\mu},\hat{\kappa})} \right)$$

as a set. The stabilizer $S_{\hat{\kappa}}$ of S at $(0, \hat{\kappa})$ with nondegenerate $\hat{\kappa}$ is given by

$$S_{\hat{\kappa}} = \{0\}.$$

And the stabilizer $S_{\hat{y}}$ of S at $(\hat{y}, 0)$ is given by

$$S_{\hat{y}} = \left\{ [\lambda, 0; 0] \,\middle|\, \lambda \in \mathbb{R}^{(m,n)} \right\} = S \;\cong\; \mathbb{R}^{(m,n)}.$$

In this section, for the present being we set $H = H_{\mathbb{R}}^{(n,m)}$ for brevity. We see that L is a closed, commutative normal subgroup of H. Since $(\lambda, \mu; \kappa) = (0, \mu; \kappa + \mu\, {}^t\lambda) \circ (\lambda, 0; 0)$ for $(\lambda, \mu; \kappa) \in H$, the homogeneous space $X = L \backslash H$ can be identified with $\mathbb{R}^{(m,n)}$ via

$$Lh = L \circ (\lambda, 0; 0) \longmapsto \lambda, \quad h = (\lambda, \mu; \kappa) \in H.$$

We observe that H acts on X by

$$(Lh) \cdot h_0 = L\, (\lambda + \lambda_0, 0; 0) = \lambda + \lambda_0,$$

where $h = (\lambda, \mu; \kappa) \in H$ and $h_0 = (\lambda_0, \mu_0; \kappa_0) \in H$.

If $h = (\lambda, \mu; \kappa) \in H$, according to the Mackey decomposition of $h = l_h \circ s_h$ with $l_h \in L$ and $s_h \in S$, (cf. [**?**]) we have

$$l_h = (0, \mu; \kappa + \mu\, {}^t\lambda), \quad s_h = (\lambda, 0; 0).$$

Thus if $h_0 = (\lambda_0, \mu_0; \kappa_0) \in H$, then we have

$$s_h \circ h_0 = (\lambda, 0; 0) \circ (\lambda_0, \mu_0; \kappa_0) = (\lambda + \lambda_0, \mu_0; \kappa_0 + \lambda\, {}^t\mu_0)$$

and so

$$l_{s_h \circ h_0} = \big(0, \mu_0; \kappa_0 + \mu_0\, {}^t\lambda_0 + \lambda\, {}^t\mu_0 + \mu_0\, {}^t\lambda\big). \tag{2.1}$$

For a real symmetric matrix $c = {}^t c \in S(m, \mathbb{R})$ with $c \neq 0$, we consider the unitary character χ_c of L defined by

$$\chi_c\left((0, \mu; \kappa)\right) = e^{\pi i\, \sigma(c\kappa)}, \quad (0, \mu; \kappa) \in L. \tag{2.2}$$

Then the representation $\mathscr{W}_c = \mathrm{Ind}_L^H\, \chi_c$ of H induced from χ_c is realized on the Hilbert space $H(\chi_c) = L^2\big(X, d\dot{h}, \mathbb{C}\big) \cong L^2\left(\mathbb{R}^{(m,n)}, d\xi\right)$ as follows. If $h_0 = (\lambda_0, \mu_0; \kappa_0) \in H$ and $x = Lh \in X$ with $h = (\lambda, \mu; \kappa) \in H$, we have

$$\left(\mathscr{W}_c(h_0) f\right)(x) = \chi_c(l_{s_h \circ h_0}) f(x h_0), \quad f \in H(\chi_c). \tag{2.3}$$

According to (2.1) and (2.2), we can describe Formula (2.3) more explicitly as follows.

$$\left[\mathscr{W}_c(h_0) f\right](\lambda) = e^{\pi i \sigma\{c(\kappa_0 + \mu_0\, {}^t\lambda_0 + 2\lambda\, {}^t\mu_0)\}}\, f(\lambda + \lambda_0), \tag{2.4}$$

where $h_0=(\lambda_0,\mu_0;\kappa_0)\in H$ and $\lambda\in\mathbb{R}^{(m,n)}$. Here we identified $x=Lh$ (resp. $xh_0=Lhh_0$) with λ (resp. $\lambda+\lambda_0$). The induced representation $\mathscr{W}_c$ is called the Schrödinger representation of H associated with χ_c. Thus $\mathscr{W}_c$ is a monomial representation.

Theorem 2.1. *Let c be a positive definite symmetric real matrix of degree m. Then the Schrödinger representation $\mathscr{W}_c$ of H is irreducible.*

Proof. The proof can be found in [**?**], Theorem 3. □

Remark 2.1. *We refer to* [**?**]-[**?**] *for more representations of the Heisenberg group* $H_{\mathbb{R}}^{(n,m)}$ *and their related topics.*

3. The Schrödinger-Weil Representation

Throughout this section we assume that $\mathcal{M}$ is a positive definite symmetric real $m\times m$ matrix. We consider the Schrödinger representation $\mathscr{W}_{\mathcal{M}}$ of the Heisenberg group $H_{\mathbb{R}}^{(n,m)}$ with the central character $\mathscr{W}_{\mathcal{M}}((0,0;\kappa))=\chi_{\mathcal{M}}((0,0;\kappa))=e^{\pi i\,\sigma(\mathcal{M}\kappa)}$, $\kappa\in S(m,\mathbb{R})$ (cf. (2.2)). We note that the symplectic group $Sp(n,\mathbb{R})$ acts on $H_{\mathbb{R}}^{(n,m)}$ by conjugation inside G^J. For a fixed element $g\in Sp(n,\mathbb{R})$, the irreducible unitary representation $\mathscr{W}_{\mathcal{M}}^g$ of $H_{\mathbb{R}}^{(n,m)}$ defined by

$$\mathscr{W}_{\mathcal{M}}^g(h)=\mathscr{W}_{\mathcal{M}}(ghg^{-1}),\quad h\in H_{\mathbb{R}}^{(n,m)} \tag{3.1}$$

has the property that

$$\mathscr{W}_{\mathcal{M}}^g((0,0;\kappa))=\mathscr{W}_{\mathcal{M}}((0,0;\kappa))=e^{\pi i\,\sigma(\mathcal{M}\kappa)}\,\mathrm{Id}_{H(\chi_{\mathcal{M}})},\quad \kappa\in S(m,\mathbb{R}).$$

Here $\mathrm{Id}_{H(\chi_{\mathcal{M}})}$ denotes the identity operator on the Hilbert space $H(\chi_{\mathcal{M}})$. According to Stone-von Neumann theorem, there exists a unitary operator $R_{\mathcal{M}}(g)$ on $H(\chi_{\mathcal{M}})$ with $R_{\mathcal{M}}(I_{2n})=\mathrm{Id}_{H(\chi_{\mathcal{M}})}$ such that

$$R_{\mathcal{M}}(g)\mathscr{W}_{\mathcal{M}}(h)=\mathscr{W}_{\mathcal{M}}^g(h)R_{\mathcal{M}}(g)\qquad\text{for all } h\in H_{\mathbb{R}}^{(n,m)}. \tag{3.2}$$

We observe that $R_{\mathcal{M}}(g)$ is determined uniquely up to a scalar of modulus one.

From now on, for brevity, we put $G=Sp(n,\mathbb{R})$. According to Schur's lemma, we have a map $c_{\mathcal{M}}:G\times G\longrightarrow T$ satisfying the relation

$$R_{\mathcal{M}}(g_1g_2)=c_{\mathcal{M}}(g_1,g_2)R_{\mathcal{M}}(g_1)R_{\mathcal{M}}(g_2)\quad\text{for all}\ \ g_1,g_2\in G. \tag{3.3}$$

We recall that T denotes the multiplicative group of complex numbers of modulus one. Therefore $R_{\mathcal{M}}$ is a projective representation of G on $H(\chi_{\mathcal{M}})$ and $c_{\mathcal{M}}$ defines the cocycle class in $H^2(G,T)$. The cocycle $c_{\mathcal{M}}$ yields the central extension $G_{\mathcal{M}}$ of G by T. The group $G_{\mathcal{M}}$ is a set $G\times T$ equipped with the following multiplication

$$(g_1,t_1)\cdot(g_2,t_2)=\big(g_1g_2,t_1t_2\,c_{\mathcal{M}}(g_1,g_2)^{-1}\big),\quad g_1,g_2\in G,\ t_1,t_2\in T. \tag{3.4}$$

We see immediately that the map $\widetilde{R}_{\mathcal{M}}:G_{\mathcal{M}}\longrightarrow GL(H(\chi_{\mathcal{M}}))$ defined by

$$\widetilde{R}_{\mathcal{M}}(g,t)=t\,R_{\mathcal{M}}(g)\quad\text{for all } (g,t)\in G_{\mathcal{M}} \tag{3.5}$$

is a *true* representation of $G_{\mathcal{M}}$. As in Section 1.7 in [**?**], we can define the map $s_{\mathcal{M}} : G \longrightarrow T$ satisfying the relation

$$c_{\mathcal{M}}(g_1,g_2)^2 = s_{\mathcal{M}}(g_1)^{-1} s_{\mathcal{M}}(g_2)^{-1} s_{\mathcal{M}}(g_1g_2) \quad \text{for all } g_1,g_2 \in G.$$

Thus we see that

$$G_{2,\mathcal{M}} = \left\{\, (g,t) \in G_{\mathcal{M}} \mid t^2 = s_{\mathcal{M}}(g)^{-1} \,\right\} \tag{3.6}$$

is the metaplectic group associated with $\mathcal{M}$ that is a two-fold covering group of G. The restriction $R_{2,\mathcal{M}}$ of $\widetilde{R}_{\mathcal{M}}$ to $G_{2,\mathcal{M}}$ is the Weil representation of G associated with $\mathcal{M}$.

If we identify $h = (\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)}$ (resp. $g \in Sp(n,\mathbb{R})$) with $(I_{2n},(\lambda,\mu;\kappa)) \in G^J$ (resp. $(g,(0,0;0)) \in G^J$), every element $\tilde{g}$ of G^J can be written as $\tilde{g} = hg$ with $h \in H_{\mathbb{R}}^{(n,m)}$ and $g \in Sp(n,\mathbb{R})$. In fact,

$$(g,(\lambda,\mu;\kappa)) = (I_{2n},((\lambda,\mu)g^{-1};\kappa))\,(g,(0,0;0)) = ((\lambda,\mu)g^{-1};\kappa)\cdot g.$$

Therefore we define the *projective* representation $\pi_{\mathcal{M}}$ of the Jacobi group G^J with cocycle $c_{\mathcal{M}}(g_1,g_2)$ by

$$\pi_{\mathcal{M}}(hg) = \mathscr{W}_{\mathcal{M}}(h)\,R_{\mathcal{M}}(g), \quad h \in H_{\mathbb{R}}^{(n,m)},\ g \in G. \tag{3.7}$$

Indeed, since $H_{\mathbb{R}}^{(n,m)}$ is a normal subgroup of G^J, for any $h_1,h_2 \in H_{\mathbb{R}}^{(n,m)}$ and $g_1,g_2 \in G$,

$$\begin{aligned}
\pi_{\mathcal{M}}(h_1g_1h_2g_2) &= \pi_{\mathcal{M}}(h_1g_1h_2g_1^{-1}g_1g_2)\\
&= \mathscr{W}_{\mathcal{M}}\big(h_1(g_1h_2g_1^{-1})\big)R_{\mathcal{M}}(g_1g_2)\\
&= c_{\mathcal{M}}(g_1,g_2)\mathscr{W}_{\mathcal{M}}(h_1)\mathscr{W}_{\mathcal{M}}^{g_1}(h_2)R_{\mathcal{M}}(g_1)R_{\mathcal{M}}(g_2)\\
&= c_{\mathcal{M}}(g_1,g_2)\mathscr{W}_{\mathcal{M}}(h_1)R_{\mathcal{M}}(g_1)\mathscr{W}_{\mathcal{M}}(h_2)R_{\mathcal{M}}(g_2)\\
&= c_{\mathcal{M}}(g_1,g_2)\pi_{\mathcal{M}}(h_1g_1)\pi_{\mathcal{M}}(h_2g_2).
\end{aligned}$$

We let

$$G_{\mathcal{M}}^J = G_{\mathcal{M}} \ltimes H_{\mathbb{R}}^{(n,m)}$$

be the semidirect product of $G_{\mathcal{M}}$ and $H_{\mathbb{R}}^{(n,m)}$ with the multiplication law

$$\begin{aligned}
&\big((g_1,t_1),(\lambda_1,\mu_1;\kappa_1)\big)\cdot\big((g_2,t_2),(\lambda_2,\mu_2\,;\kappa_2)\big)\\
=\ &\big((g_1,t_1)(g_2,t_2),(\tilde{\lambda}+\lambda_2,\tilde{\mu}+\mu_2\,;\kappa_1+\kappa_2+\tilde{\lambda}\,{}^t\mu_2-\tilde{\mu}\,{}^t\lambda_2)\big),
\end{aligned}$$

where $(g_1,t_1),(g_2,t_2) \in G_{\mathcal{M}},\ (\lambda_1,\mu_1;\kappa_1),(\lambda_2,\mu_2\,;\kappa_2) \in H_{\mathbb{R}}^{(n,m)}$ and $(\tilde{\lambda},\tilde{\mu}) = (\lambda,\mu)g_2$. If we identify $h = (\lambda,\mu\,;\kappa) \in H_{\mathbb{R}}^{(n,m)}$ (resp. $(g,t) \in G_{\mathcal{M}}$) with $((I_{2n},1),(\lambda,\mu\,;\kappa)) \in G_{\mathcal{M}}^J$ (resp. $((g,t),(0,0;0)) \in G_{\mathcal{M}}^J$), we see easily that every element $\big((g,t),(\lambda,\mu\,;\kappa)\big)$ of $G_{\mathcal{M}}^J$ can be expressed as

$$\big((g,t),(\lambda,\mu\,;\kappa)\big) = \big((I_{2n},1),((\lambda,\mu)g^{-1};\kappa)\big)\big((g,t),(0,0;0)\big) = ((\lambda,\mu)g^{-1};\kappa)(g,t).$$

Now we can define the *true* representation $\widetilde{\omega}_{\mathcal{M}}$ of $G_{\mathcal{M}}^J$ by

$$\widetilde{\omega}_{\mathcal{M}}(h\cdot(g,t)) = t\,\pi_{\mathcal{M}}(hg) = t\,\mathscr{W}_{\mathcal{M}}(h)\,R_{\mathcal{M}}(g), \quad h \in H_{\mathbb{R}}^{(n,m)},\ (g,t) \in G_{\mathcal{M}}. \tag{3.8}$$

Indeed, since $H_{\mathbb{R}}^{(n,m)}$ is a normal subgroup of $G^J_{\mathcal{M}}$,

$$\begin{aligned}
&\widetilde{\omega}_{\mathcal{M}}\big(h_1(g_1,t_1)h_2(g_2,t_2)\big)\\
=\ &\widetilde{\omega}_{\mathcal{M}}\big(h_1(g_1,t_1)h_2(g_1,t_1)^{-1}(g_1,t_1)(g_2,t_2)\big)\\
=\ &\widetilde{\omega}_{\mathcal{M}}\big(h_1(g_1,t_1)h_2(g_1,t_1)^{-1}\big(g_1g_2,t_1t_2\,c_{\mathcal{M}}(g_1,g_2)^{-1}\big)\big)\\
=\ &t_1t_2\,c_{\mathcal{M}}(g_1,g_2)^{-1}\,\mathscr{W}_{\mathcal{M}}\big(h_1(g_1,t_1)h_2(g_1,t_1)^{-1}\big)R_{\mathcal{M}}(g_1g_2)\\
=\ &t_1t_2\,\mathscr{W}_{\mathcal{M}}(h_1)\mathscr{W}_{\mathcal{M}}\big((g_1,t_1)h_2(g_1,t_1)^{-1}\big)\,R_{\mathcal{M}}(g_1)R_{\mathcal{M}}(g_2)\\
=\ &t_1t_2\,\mathscr{W}_{\mathcal{M}}(h_1)\mathscr{W}_{\mathcal{M}}\big(g_1h_2g_1^{-1}\big)\,R_{\mathcal{M}}(g_1)R_{\mathcal{M}}(g_2)\\
=\ &t_1t_2\,\mathscr{W}_{\mathcal{M}}(h_1)R_{\mathcal{M}}(g_1)\,\mathscr{W}_{\mathcal{M}}(h_2)R_{\mathcal{M}}(g_2)\\
=\ &\{t_1\,\pi_{\mathcal{M}}(h_1g_1)\}\,\{t_2\,\pi_{\mathcal{M}}(h_2g_2)\}\\
=\ &\widetilde{\omega}_{\mathcal{M}}\big(h_1(g_1,t_1)\big)\,\tilde{\omega}_{\mathcal{M}}\big(h_2(g_2,t_2)\big).
\end{aligned}$$

Here we used the fact that $(g_1,t_1)h_2(g_1,t_1)^{-1}=g_1h_2g_1^{-1}$.

We recall that the following matrices

$$\begin{aligned}
t(b)\ &=\ \begin{pmatrix} I_n & b\\ 0 & I_n\end{pmatrix}\ \text{with any } b={}^tb\in\mathbb{R}^{(n,n)},\\
g(\alpha)\ &=\ \begin{pmatrix} {}^t\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}\ \text{with any } \alpha\in GL(n,\mathbb{R}),\\
\sigma_n\ &=\ \begin{pmatrix} 0 & -I_n\\ I_n & 0\end{pmatrix}
\end{aligned}$$

generate the symplectic group $G=Sp(n,\mathbb{R})$ (cf. [?, p. 326], [?, p. 210]). Therefore the following elements $h_t(\lambda,\mu\,;\kappa)$, $t(b\,;t)$, $g(\alpha\,;t)$ and $\sigma_{n\,;t}$ of $G_{\mathcal{M}}\ltimes H_{\mathbb{R}}^{(n,m)}$ defined by

$$\begin{aligned}
&h_t(\lambda,\mu\,;\kappa)=\big((I_{2n},t),(\lambda,\mu;\kappa)\big)\ \text{with } t\in T,\ \lambda,\mu\in\mathbb{R}^{(m,n)} \text{ and } \kappa\in\mathbb{R}^{(m,m)},\\
&t(b\,;t)=\big((t(b),t),(0,0;0)\big)\ \text{with any } b={}^tb\in\mathbb{R}^{(n,n)},\ t\in T,\\
&g(\alpha\,;t)=\big((g(\alpha),t),(0,0;0)\big)\ \text{with any } \alpha\in GL(n,\mathbb{R}) \text{ and } t\in T,\\
&\sigma_{n\,;t}=((\sigma_n,t),(0,0;0))\ \text{with } t\in T
\end{aligned}$$

generate the group $G_{\mathcal{M}}\ltimes H_{\mathbb{R}}^{(n,m)}$. We can show that the representation $\widetilde{\omega}_{\mathcal{M}}$ is realized on the representation $H(\chi_{\mathcal{M}})=L^2\big(\mathbb{R}^{(m,n)}\big)$ as follows: for each $f\in L^2\big(\mathbb{R}^{(m,n)}\big)$ and $x\in\mathbb{R}^{(m,n)}$, the actions of $\widetilde{\omega}_{\mathcal{M}}$ on the generators are given by

$$\big[\widetilde{\omega}_{\mathcal{M}}\big(h_t(\lambda,\mu\,;\kappa)\big)f\big]\,(x)\ =\ t\,e^{\pi i\,\sigma\{\mathcal{M}(\kappa+\mu\,{}^t\lambda+2\,x\,{}^t\mu)\}}\,f(x+\lambda), \tag{3.9}$$

$$\big[\widetilde{\omega}_{\mathcal{M}}\big(t(b\,;t)\big)f\big]\,(x)\ =\ t\,e^{\pi i\,\sigma(\mathcal{M}\,x\,b\,{}^tx)}f(x), \tag{3.10}$$

$$\big[\widetilde{\omega}_{\mathcal{M}}\big(g(\alpha\,;t)\big)f\big]\,(x)\ =\ t\,|\det\alpha|^{\frac{m}{2}}\,f(x\,{}^t\alpha), \tag{3.11}$$

$$\big[\widetilde{\omega}_{\mathcal{M}}\big(\sigma_{n\,;t}\big)f\big]\,(x)=t\,\big(\det\mathcal{M}\big)^{\frac{n}{2}}\int_{\mathbb{R}^{(m,n)}} f(y)\,e^{-2\,\pi i\,\sigma(\mathcal{M}\,y\,{}^tx)}\,dy. \tag{3.12}$$

Let

$$G^J_{2,\mathcal{M}}=G_{2,\mathcal{M}}\ltimes H_{\mathbb{R}}^{(n,m)}$$

be the semidirect product of $G_{2,\mathcal{M}}$ and $H_{\mathbb{R}}^{(n,m)}$. Then $G_{2,\mathcal{M}}^J$ is a subgroup of $G_{\mathcal{M}}^J$ which is a two-fold covering group of the Jacobi group G^J. The restriction $\omega_{\mathcal{M}}$ of $\widetilde{\omega}_{\mathcal{M}}$ to $G_{2,\mathcal{M}}^J$ is called the Schrödinger-Weil representation of G^J associated with $\mathcal{M}$.

We denote by $L_+^2\big(\mathbb{R}^{(m,n)}\big)$ $\big($resp. $L_-^2\big(\mathbb{R}^{(m,n)}\big)\big)$ the subspace of $L^2\big(\mathbb{R}^{(m,n)}\big)$ consisting of even (resp. odd) functions in $L^2\big(\mathbb{R}^{(m,n)}\big)$. According to Formulas (3.10)–(3.12), $R_{2,\mathcal{M}}$ is decomposed into representations of $R_{2,\mathcal{M}}^{\pm}$

$$R_{2,\mathcal{M}} = R_{2,\mathcal{M}}^{+} \oplus R_{2,\mathcal{M}}^{-},$$

where $R_{2,\mathcal{M}}^{+}$ and $R_{2,\mathcal{M}}^{-}$ are the even Weil representation and the odd Weil representation of G that are realized on $L_+^2\big(\mathbb{R}^{(m,n)}\big)$ and $L_-^2\big(\mathbb{R}^{(m,n)}\big)$ respectively. Obviously the center $\mathscr{Z}_{2,\mathcal{M}}^J$ of $G_{2,\mathcal{M}}^J$ is given by

$$\mathscr{Z}_{2,\mathcal{M}}^J = \big\{ \big((I_{2n},1),(0,0;\kappa)\big) \in G_{2,\mathcal{M}}^J \big\} \cong S(m,\mathbb{R}).$$

We note that the restriction of $\omega_{\mathcal{M}}$ to $G_{2,\mathcal{M}}$ coincides with $R_{2,\mathcal{M}}$ and $\omega_{\mathcal{M}}(h) = \mathscr{W}_{\mathcal{M}}(h)$ for all $h \in H_{\mathbb{R}}^{(n,m)}$.

Remark 3.1. *In the case* $n = m = 1$, $\omega_{\mathcal{M}}$ *is dealt in* [?] *and* [?]. *We refer to* [?] *and* [?] *for more details about the Weil representation* $R_{2,\mathcal{M}}$.

Remark 3.2. *The Schrödinger-Weil representation is applied usefully to the theory of Maass-Jacobi forms* [?].

4. Theta Sums

Let $\mathcal{M}$ be a positive definite symmetric real matrix of degree m. We recall the Schrödinger representation $\mathscr{W}_{\mathcal{M}}$ of the Heisenberg group $H_{\mathbb{R}}^{(n,m)}$ associate with $\mathcal{M}$ given by Formula (2.4) in Section 2. We note that for an element $(\lambda,\mu;\kappa)$ of $H_{\mathbb{R}}^{(n,m)}$, we have the decomposition

$$(\lambda,\mu;\kappa) = (\lambda,0;0) \circ (0,\mu;0) \circ (0,0;\kappa - \lambda\, {}^t\mu).$$

We consider the embedding $\Phi_n : SL(2,\mathbb{R}) \longrightarrow Sp(n,\mathbb{R})$ defined by

$$\Phi_n\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) := \begin{pmatrix} aI_n & bI_n \\ cI_n & dI_n \end{pmatrix}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}). \tag{4.1}$$

For $x, y \in \mathbb{R}^{(m,n)}$, we put

$$(x,y)_{\mathcal{M}} := \sigma({}^t x \mathcal{M} y) \qquad \text{and} \qquad \|x\|_{\mathcal{M}} := \sqrt{(x,x)_{\mathcal{M}}}.$$

According to Formulas (3.10)-(3.12), for any $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}) \hookrightarrow Sp(n,\mathbb{R})$ and $f \in L^2\left(\mathbb{R}^{(m,n)}\right)$, we have the following explicit representation

$$[R_{\mathcal{M}}(M)f](x) = \begin{cases} |a|^{\frac{mn}{2}} e^{ab\|x\|_{\mathcal{M}}^2 \pi i} f(ax) & \text{if } c = 0, \\ (\det \mathcal{M})^{\frac{n}{2}}\, |c|^{-\frac{mn}{2}} \int_{\mathbb{R}^{(m,n)}} e^{\frac{\alpha(M,x,y,\mathcal{M})}{c}\pi i} f(y)dy & \text{if } c \neq 0, \end{cases} \tag{4.2}$$

where

$$\alpha(M,x,y,\mathcal{M}) = a\,\|x\|^2_{\mathcal{M}} + d\,\|y\|^2_{\mathcal{M}} - 2(x,y)_{\mathcal{M}}.$$

Indeed, if $a=0$ and $c \neq 0$, using the decomposition

$$M = \begin{pmatrix} 0 & -c^{-1} \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}$$

and if $a \neq 0$ and $c \neq 0$, using the decomposition

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c^{-1} \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} ac & ad \\ 0 & (ac)^{-1} \end{pmatrix},$$

we obtain Formula (4.2).

If

$$M_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \quad \text{and} \quad M_3 = \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} \in SL(2,\mathbb{R})$$

with $M_3 = M_1M_2$, the corresponding cocycle is given by

$$c_{\mathcal{M}}(M_1,M_2) = e^{-i\,\pi\,mn\,\mathrm{sign}(c_1c_2c_3)/4}, \tag{4.3}$$

where

$$\mathrm{sign}(x) = \begin{cases} -1 & (x<0) \\ 0 & (x=0) \\ 1 & (x>0). \end{cases}$$

In the special case when

$$M_1 = \begin{pmatrix} \cos\phi_1 & -\sin\phi_1 \\ \sin\phi_1 & \cos\phi_1 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} \cos\phi_2 & -\sin\phi_2 \\ \sin\phi_2 & \cos\phi_2 \end{pmatrix},$$

we find

$$c_{\mathcal{M}}(M_1,M_2) = e^{-i\,\pi\,mn\,(\sigma_{\phi_1}+\sigma_{\phi_2}-\sigma_{\phi_1+\phi_2})/4},$$

where

$$\sigma_\phi = \begin{cases} 2\nu & \text{if } \phi = \nu\pi \\ 2\nu+1 & \text{if } \nu\pi < \phi < (\nu+1)\pi. \end{cases}$$

It is well known that every $M \in SL(2,\mathbb{R})$ admits the unique Iwasawa decomposition

$$M = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^{1/2} & 0 \\ 0 & v^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}, \tag{4.4}$$

where $\tau = u+iv \in \mathbb{H}_1$ and $\phi \in [0,2\pi)$. This parametrization $M=(\tau,\phi)$ in $SL(2,\mathbb{R})$ leads to the natural action of $SL(2,\mathbb{R})$ on $\mathbb{H}_1 \times [0,2\pi)$ defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau,\phi) := \left(\frac{a\tau+b}{c\tau+d},\ \phi + \arg(c\tau+d) \bmod 2\pi \right). \tag{4.5}$$

Lemma 4.1. *For two elements g_1 and g_2 in $SL(2,\mathbb{R})$, we let*

$$g_1 = \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1^{1/2} & 0 \\ 0 & v_1^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\phi_1 & -\sin\phi_1 \\ \sin\phi_1 & \cos\phi_1 \end{pmatrix}$$

and

$$g_2 = \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_2^{1/2} & 0 \\ 0 & v_2^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\phi_2 & -\sin\phi_2 \\ \sin\phi_2 & \cos\phi_2 \end{pmatrix}$$

be the Iwasawa decompositions of g_1 *and* g_2 *respectively, where* $u_1, u_2 \in \mathbb{R},\ v_1 > 0,\ v_2 > 0$ *and* $0 \le \phi_1, \phi_2 < 2\pi$. *Let*

$$g_3 = g_1 g_2 = \begin{pmatrix} 1 & u_3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_3^{1/2} & 0 \\ 0 & v_3^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\phi_3 & -\sin\phi_3 \\ \sin\phi_3 & \cos\phi_3 \end{pmatrix}$$

be the Iwasawa decomposition of $g_3 = g_1 g_2$. *Then we have*

$$\begin{aligned} u_3 &= \frac{A}{(u_2\sin\phi_1 + \cos\phi_1)^2 + (v_2\sin\phi_1)^2}, \\ v_3 &= \frac{v_1 v_2}{(u_2\sin\phi_1 + \cos\phi_1)^2 + (v_2\sin\phi_1)^2} \end{aligned}$$

and

$$\phi_3 = tan^{-1}\left[\frac{(v_2\cos\phi_2 + u_2\sin\phi_2)\tan\phi_1 + \sin\phi_2}{(-v_2\sin\phi_2 + u_2\cos\phi_2)\tan\phi_1 + \cos\phi_2} \right],$$

where

$$\begin{aligned} A \;=\; & u_1(u_2\sin\phi_1 + \cos\phi_1)^2 + (u_1v_2 - v_1u_2)\sin^2\phi_1 \\ & + v_1u_2\cos^2\phi_1 + v_1(u_2^2 + v_2^2 - 1)\sin\phi_1\cos\phi_1. \end{aligned}$$

Proof. If $g \in SL(2,\mathbb{R})$ has the unique Iwasawa decomposition (4.4), then we get the following

$$\begin{aligned} a &= v^{1/2}\cos\phi + uv^{-1/2}\sin\phi, \\ b &= -v^{1/2}\sin\phi + uv^{-1/2}\cos\phi, \\ c &= v^{-1/2}\sin\phi, \quad d = v^{-1/2}\cos\phi, \\ u &= (ac+bd)\left(c^2+d^2\right)^{-1}, \quad v = \left(c^2+d^2\right)^{-1}, \quad \tan\phi = \frac{c}{d}. \end{aligned}$$

We set

$$g_3 = g_1 g_2 = \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}.$$

Since

$$u_3 = (a_3c_3 + b_3d_3)\left(c_3^2 + d_3^2\right)^{-1}, \quad v = \left(c_3^2 + d_3^2\right)^{-1}, \quad \tan\phi_3 = \frac{c_3}{d_3},$$

by an easy computation, we obtain the desired results. □

Now we use the new coordinates $(\tau = u + iv, \phi)$ with $\tau \in \mathbb{H}_1$ and $\phi \in [0, 2\pi)$ in $SL(2,\mathbb{R})$. According to Formulas (3.10)-(3.12), the projective representation $R_{\mathcal{M}}$ of $SL(2,\mathbb{R}) \hookrightarrow Sp(n,\mathbb{R})$ reads in these coordinates $(\tau = u + iv, \phi)$ as follows:

$$[R_{\mathcal{M}}(\tau,\phi)f](x) = v^{\frac{mn}{4}}\, e^{u\|x\|^2_{\mathcal{M}}\pi i}\, [R_{\mathcal{M}}(i,\phi)f]\left(v^{1/2}x\right), \tag{4.6}$$

where $f \in L^2\left(\mathbb{R}^{(m,n)}\right)$, $x \in \mathbb{R}^{(m,n)}$ and

$$\begin{aligned} &[R_{\mathcal{M}}(i,\phi)f](x) \\ &= \begin{cases} f(x) & \text{if } \phi \equiv 0 \text{ mod } 2\pi, \\ f(-x) & \text{if } \phi \equiv \pi \text{ mod } 2\pi, \\ (\det\mathcal{M})^{\frac{n}{2}}\, |\sin\phi|^{-\frac{mn}{2}} \int_{\mathbb{R}^{(m,n)}} e^{B(x,y,\phi,\mathcal{M})\pi i} f(y)dy & \text{if } \phi \not\equiv 0 \text{ mod } \pi. \end{cases} \end{aligned} \tag{4.7}$$

Here

$$B(x,y,\phi,\mathcal{M})=\frac{\left(\|x\|^2_{\mathcal{M}}+\|y\|^2_{\mathcal{M}}\right)\cos\phi-2(x,y)_{\mathcal{M}}}{\sin\phi}.$$

Now we set

$$S=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

We note that

$$\left[R_{\mathcal{M}}\left(i,\frac{\pi}{2}\right)f\right](x)=\left[R_{\mathcal{M}}(S)f\right](x)=(\det\mathcal{M})^{\frac{n}{2}}\int_{\mathbb{R}^{(m,n)}}f(y)\,e^{-2\,(x,\,y)_{\mathcal{M}}\,\pi\,i}\,dy \tag{4.8}$$

for $f\in L^2\left(\mathbb{R}^{(m,n)}\right)$.

Remark 4.1. *For Schwartz functions $f\in\mathscr{S}\left(\mathbb{R}^{(m,n)}\right)$, we have*

$$\lim_{\phi\longrightarrow 0\pm}|\sin\phi|^{-\frac{mn}{2}}\int_{\mathbb{R}^{(m,n)}}e^{B(x,y,\phi,\mathcal{M})\,\pi\,i}\,f(y)dy=e^{\pm i\,\pi\,mn/4}f(x)\neq f(x).$$

Therefore the projective representation $R_{\mathcal{M}}$ is not continuous at $\phi=\nu\pi\ (\nu\in\mathbb{Z})$ in general. If we set

$$\tilde{R}_{\mathcal{M}}(\tau,\phi)=e^{-i\,\pi\,mn\sigma_{\phi}/4}R_{\mathcal{M}}(\tau,\phi),$$

*$\tilde{R}_{\mathcal{M}}$ corresponds to a unitary representation of the double cover of $SL(2,\mathbb{R})$ (cf. (3.5) and [**?**]). This means in particular that*

$$\tilde{R}_{\mathcal{M}}(i,\phi)\tilde{R}_{\mathcal{M}}(i,\phi')=\tilde{R}_{\mathcal{M}}(i,\phi+\phi'),$$

where $\phi\in[0,4\pi)$ parametrises the double cover of $SO(2)\subset SL(2,\mathbb{R})$.

We observe that for any element $(g,(\lambda,\mu;\kappa))\in G^J$ with $g\in Sp(n,\mathbb{R})$ and $(\lambda,\mu;\kappa)\in H^{(n,m)}_{\mathbb{R}}$, we have the following decomposition

$$(g,(\lambda,\mu;\kappa))=(I_{2n},((\lambda,\mu)g^{-1};\kappa))\,(g,(0,0;0))=((\lambda,\mu)g^{-1};\kappa)\cdot g.$$

Thus $Sp(n,\mathbb{R})$ acts on $H^{(n,m)}_{\mathbb{R}}$ naturally by

$$g\cdot(\lambda,\mu;\kappa)=\left((\lambda,\mu)g^{-1};\kappa\right),\qquad g\in Sp(n,\mathbb{R}),\ (\lambda,\mu;\kappa)\in H^{(n,m)}_{\mathbb{R}}.$$

Definition 4.1. *For any Schwartz function $f\in\mathscr{S}\left(\mathbb{R}^{(m,n)}\right)$, we define the function $\Theta^{[\mathcal{M}]}_f$ on the Jacobi group $SL(2,\mathbb{R})\ltimes H^{(n,m)}_{\mathbb{R}}\hookrightarrow G^J$ by*

$$\Theta^{[\mathcal{M}]}_f(\tau,\phi\,;\lambda,\mu,\kappa):=\sum_{\omega\in\mathbb{Z}^{(m,n)}}\left[\pi_{\mathcal{M}}\left((\lambda,\mu;\kappa)(\tau,\phi)\right)f\right](\omega), \tag{4.9}$$

where $(\tau,\phi)\in SL(2,\mathbb{R})$ and $(\lambda,\mu\,;\kappa)\in H^{(n,m)}_{\mathbb{R}}$. The projective representation $\pi_{\mathcal{M}}$ of the Jacobi group G^J was already defined by Formula (3.7). More precisely, for $\tau=u+iv\in\mathbb{H}_1$ and $(\lambda,\mu;\kappa)\in H^{(n,m)}_{\mathbb{R}}$, we have

$$\begin{aligned}\Theta^{[\mathcal{M}]}_f(\tau,\phi\,;\lambda,\mu,\kappa)&=v^{\frac{mn}{4}}\,e^{2\,\pi\,i\,\sigma(\mathcal{M}(\kappa+\mu^t\lambda))}\\ &\times\sum_{\omega\in\mathbb{Z}^{(m,n)}}e^{\pi\,i\,\left\{u\|\omega+\lambda\|^2_{\mathcal{M}}+2(\omega,\,\mu)_{\mathcal{M}}\right\}}\left[R_{\mathcal{M}}(i,\phi)f\right]\left(v^{1/2}(\omega+\lambda)\right).\end{aligned}$$

Lemma 4.2. *We set* $f_\phi := \tilde{R}_{\mathcal{M}}(i,\phi)f$ *for* $f \in \mathscr{S}\left(\mathbb{R}^{(m,n)}\right)$. *Then for any* $R>1$, *there exists a constant* C_R *such that for all* $x \in \mathbb{R}^{(m,n)}$ *and* $\phi \in \mathbb{R}$,

$$|f_\phi(x)| \leq C_R\,\left(1+\|x\|_{\mathcal{M}}\right)^{-R}.$$

Proof. Following the arguments in the proof of Lemma 4.3 in [**?**], pp. 428-429, we get the desired result. □

Theorem 4.1 (Jacobi 1)**.** *Let* $\mathcal{M}$ *be a positive definite symmetric integral matrix of degree* m *such that* $\mathcal{M}\mathbb{Z}^{(m,n)} = \mathbb{Z}^{(m,n)}$. *Then for any Schwartz function* $f \in \mathscr{S}\left(\mathbb{R}^{(m,n)}\right)$, *we have*

$$\Theta_f^{[\mathcal{M}]}\left(-\frac{1}{\tau},\,\phi + arg\,\tau\,;-\mu,\lambda,\kappa\right) = \left(\det\mathcal{M}\right)^{-\frac{n}{2}}\,c_{\mathcal{M}}(S,(\tau,\phi))\,\Theta_f^{[\mathcal{M}]}(\tau,\phi\,;\lambda,\mu,\kappa),$$

where

$$c_{\mathcal{M}}(S,(\tau,\phi)) := e^{i\,\pi mn\,sign(\sin\phi\,\sin(\phi+\arg\tau))}.$$

Proof. First we recall that for any Schwartz function $\varphi \in \mathscr{S}\left(\mathbb{R}^{(m,n)}\right)$, the Fourier transform $\mathscr{F}\varphi$ of φ is given by

$$\left(\mathscr{F}\varphi\right)(x) = \int_{\mathbb{R}^{(m,n)}} \varphi(y)\,e^{-2\pi i\,\sigma(y\,{}^t x)}dy.$$

Now we put

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL(2,\mathbb{Z}) \hookrightarrow Sp(n,\mathbb{R})$$

and for any $F \in \mathscr{S}\left(\mathbb{R}^{(m,n)}\right)$, we put

$$F_{\mathcal{M}}(x) := F(\mathcal{M}^{-1}x), \quad x \in \mathbb{R}^{(m,n)}.$$

According to Formula (3.12), for any $F \in \mathscr{S}\left(\mathbb{R}^{(m,n)}\right)$,

$$\begin{aligned}
\left[R_{\mathcal{M}}(S)F\right](x) &= \left(\det\mathcal{M}\right)^{\frac{n}{2}} \int_{\mathbb{R}^{(m,n)}} F(y)\,e^{-2\pi i\,\sigma(\mathcal{M}y\,{}^t x)}dy \\
&= \left(\det\mathcal{M}\right)^{-\frac{n}{2}} \int_{\mathbb{R}^{(m,n)}} F(\mathcal{M}^{-1}y)\,e^{-2\pi i\,\sigma(y\,{}^t x)}dy \\
&= \left(\det\mathcal{M}\right)^{-\frac{n}{2}} \int_{\mathbb{R}^{(m,n)}} F_{\mathcal{M}}(y)\,e^{-2\pi i\,\sigma(y\,{}^t x)}dy \\
&= \left(\det\mathcal{M}\right)^{-\frac{n}{2}}\,\left[\mathscr{F}F_{\mathcal{M}}\right](x).
\end{aligned}$$

Thus we have

$$\mathscr{F}F_{\mathcal{M}} = \left(\det\mathcal{M}\right)^{\frac{n}{2}}\,R_{\mathcal{M}}(S)F \qquad \text{for } F \in \mathscr{S}\left(\mathbb{R}^{(m,n)}\right). \tag{4.10}$$

By Lemma 4.1, we get easily

$$S\cdot(\tau,\phi) = \left(-\frac{1}{\tau},\phi+\arg\tau\right). \tag{4.11}$$

If we take $F = \pi_{\mathcal{M}}((\lambda,\mu\,;\kappa)(\tau,\phi))f$ for $f \in \mathscr{S}\left(\mathbb{R}^{(m,n)}\right)$, a fixed element $(\lambda,\mu\,;\kappa) \in H_{\mathbb{R}}^{(n,m)}$ and an fixed element $(\tau,\phi) \in SL(2,\mathbb{R})$, then it is easily seen that $F \in \mathscr{S}\left(\mathbb{R}^{(m,n)}\right)$.

According to Formulas (4.11), if we take $F = \pi_{\mathcal{M}}((\lambda,\mu\,;\kappa)(\tau,\phi))f$ for $f \in \mathscr{S}\left(\mathbb{R}^{(m,n)}\right)$,

$$\begin{aligned}
\big[R_{\mathcal{M}}(S)F\big](x) &= \big[R_{\mathcal{M}}(S)\pi_{\mathcal{M}}\big((\lambda,\mu\,;\kappa)(\tau,\phi)\big)f\big](x), \quad x \in \mathbb{R}^{(m,n)} \\
&= \left[R_{\mathcal{M}}(S)\mathscr{W}_{\mathcal{M}}(\lambda,\mu\,;\kappa)R_{\mathcal{M}}(\tau,\phi)f\right](x) \\
&= \big[\mathscr{W}_{\mathcal{M}}\big((\lambda,\mu)S^{-1};\kappa\big)R_{\mathcal{M}}(S)R_{\mathcal{M}}(\tau,\phi)f\big](x) \\
&= c_{\mathcal{M}}(S,(\tau,\phi))^{-1}\,\big[\mathscr{W}_{\mathcal{M}}(-\mu,\lambda\,;\kappa)R_{\mathcal{M}}\big(S\cdot(\tau,\phi)\big)f\big](x) \\
&= c_{\mathcal{M}}(S,(\tau,\phi))^{-1}\left[\mathscr{W}_{\mathcal{M}}(-\mu,\lambda\,;\kappa)R_{\mathcal{M}}\left(-\frac{1}{\tau},\phi+\arg\tau\right)f\right](x) \\
&= c_{\mathcal{M}}(S,(\tau,\phi))^{-1}\left[\pi_{\mathcal{M}}\left((-\mu,\lambda\,;\kappa)\left(-\frac{1}{\tau},\phi+\arg\tau\right)\right)f\right](x).
\end{aligned}$$

Thus we obtain

$$\big[R_{\mathcal{M}}(S)F\big](x) = c_{\mathcal{M}}(S,(\tau,\phi))^{-1}\left[\pi_{\mathcal{M}}\left((-\mu,\lambda\,;\kappa)\left(-\frac{1}{\tau},\phi+\arg\tau\right)\right)f\right](x). \tag{4.12}$$

According to Poisson summation formula, we have

$$\sum_{\omega\in\mathbb{Z}^{(m,n)}}\left[\mathscr{F}F_{\mathcal{M}}\right](\omega) = \sum_{\omega\in\mathbb{Z}^{(m,n)}} F_{\mathcal{M}}(\omega). \tag{4.13}$$

It follows from (4.10) and (4.12) that

$$\begin{aligned}
\sum_{\omega\in\mathbb{Z}^{(m,n)}}\left[\mathscr{F}F_{\mathcal{M}}\right](\omega) &= \big(\det\mathcal{M}\big)^{\frac{n}{2}}\sum_{\omega\in\mathbb{Z}^{(m,n)}}\big[R_{\mathcal{M}}(S)F\big](\omega) \\
&= \big(\det\mathcal{M}\big)^{\frac{n}{2}}\,c_{\mathcal{M}}(S,(\tau,\phi))^{-1} \\
&\quad\times\sum_{\omega\in\mathbb{Z}^{(m,n)}}\left[\pi_{\mathcal{M}}\left((-\mu,\lambda\,;\kappa)\left(-\frac{1}{\tau},\phi+\arg\tau\right)\right)f\right](x) \\
&= \big(\det\mathcal{M}\big)^{\frac{n}{2}}\,c_{\mathcal{M}}(S,(\tau,\phi))^{-1}\,\Theta_f^{[\mathcal{M}]}\left(-\frac{1}{\tau},\,\phi+\arg\tau\,;-\mu,\lambda,\kappa\right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\sum_{\omega\in\mathbb{Z}^{(m,n)}} F_{\mathcal{M}}(\omega) &= \sum_{\omega\in\mathbb{Z}^{(m,n)}} F\big(\mathcal{M}^{-1}\omega\big) \\
&= \sum_{\omega\in\mathbb{Z}^{(m,n)}}\left[\pi_{\mathcal{M}}((\lambda,\mu\,;\kappa)(\tau,\phi))f\right]\big(\mathcal{M}^{-1}\omega\big) \\
&= \sum_{\omega\in\mathbb{Z}^{(m,n)}}\left[\pi_{\mathcal{M}}((\lambda,\mu\,;\kappa)(\tau,\phi))f\right](\omega) \qquad \Big(\because\ \mathcal{M}^{-1}\mathbb{Z}^{(m,n)} = \mathbb{Z}^{(m,n)}\Big) \\
&= \Theta_f^{[\mathcal{M}]}(\tau,\phi\,;\lambda,\mu,\kappa).
\end{aligned}$$

Hence from (4.13) we obtain the desired formula

$$\Theta_f^{[\mathcal{M}]}\left(-\frac{1}{\tau},\,\phi+\arg\tau\,;-\mu,\lambda,\kappa\right) = \big(\det\mathcal{M}\big)^{-\frac{n}{2}}\,c_{\mathcal{M}}(S,(\tau,\phi))\,\Theta_f^{[\mathcal{M}]}(\tau,\phi\,;\lambda,\mu,\kappa).$$

If

$$S = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \quad (\tau,\phi) = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \quad\text{and}\quad S\cdot(\tau,\phi) = \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} \in SL(2,\mathbb{R}),$$

according to Lemma 4.1, we get easily

$$c_1c_2c_3 = \big(u^2+v^2\big)^{1/2}\sin\phi\,\sin(\phi+\arg\tau),$$

where

$$(\tau,\phi) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^{1/2} & 0 \\ 0 & v^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

is the Iwasawa decomposition of $(\tau,\phi)\in SL(2,\mathbb{R})$. Thus we obtain

$$c_{\mathcal{M}}(S,(\tau,\phi)) = e^{i\,\pi mn\,\mathrm{sign}(c_1c_2c_3)} = e^{i\,\pi mn\,\mathrm{sign}(\sin\phi\,\sin(\phi+\arg\tau))}.$$

This completes the proof. □

Theorem 4.2 (Jacobi 2)**.** *Let* $\mathcal{M}=(\mathcal{M}_{kl})$ *be a positive definite symmetric integral* $m\times m$ *matrix and let* $s=(s_{kj})\in\mathbb{Z}^{(m,n)}$ *be integral. Then we have*

$$\Theta_f^{[\mathcal{M}]}(\tau+2,\phi\,;\lambda,s-2\,\lambda+\mu,\kappa-s\,{}^t\lambda) = \Theta_f^{[\mathcal{M}]}(\tau,\phi\,;\lambda,\mu,\kappa)$$

for all $(\tau,\phi)\in SL(2,\mathbb{R})$ *and* $(\lambda,\mu;\kappa)\in H_{\mathbb{R}}^{(n,m)}$.

Proof. For brevity, we put $T_*=\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. According to Lemma 4.1, for any $(\tau,\phi)\in SL(2,\mathbb{R})$, the multiplication of T_* and (τ,ϕ) is given by

$$T_*(\tau,\phi) = (\tau+2,\phi). \tag{4.14}$$

For $s\in\mathbb{R}^{(m,n)}$, $(\lambda,\mu\,;\kappa)\in H_{\mathbb{R}}^{(n,m)}$ and $(\tau,\phi)\in SL(2,\mathbb{R})$, according to (4.14),

$$\begin{aligned}
& \pi_{\mathcal{M}}((0,s;0)T_*)\,\pi_{\mathcal{M}}((\lambda,\mu\,;\kappa)(\tau,\phi)) \\
= \;& \mathscr{W}_{\mathcal{M}}(0,s;0)R_{\mathcal{M}}(T_*)\mathscr{W}_{\mathcal{M}}(\lambda,\mu\,;\kappa)R_{\mathcal{M}}(\tau,\phi) \\
= \;& \mathscr{W}_{\mathcal{M}}(0,s;0)\mathscr{W}_{\mathcal{M}}\big((\lambda,\mu)T_*^{-1}\,;\kappa\big)R_{\mathcal{M}}(T_*)R_{\mathcal{M}}(\tau,\phi) \\
= \;& c_{\mathcal{M}}(T_*,(\tau,\phi))^{-1}\mathscr{W}_{\mathcal{M}}(\lambda,s-2\,\lambda+\mu\,;\kappa-s\,{}^t\lambda)R_{\mathcal{M}}\big(T_*(\tau,\phi)\big) \\
= \;& \mathscr{W}_{\mathcal{M}}(\lambda,s-2\,\lambda+\mu\,;\kappa-s\,{}^t\lambda)R_{\mathcal{M}}(\tau+2,\phi) \\
= \;& \pi_{\mathcal{M}}\big((\lambda,s-2\,\lambda+\mu\,;\kappa-s\,{}^t\lambda)(\tau+2,\phi)\big).
\end{aligned}$$

Here we used the fact that $c_{\mathcal{M}}(T_*,(\tau,\phi))=1$ because T_* is upper triangular.

On the other hand, according to the assumptions on $\mathcal{M}$ and s, for $f\in\mathscr{S}\left(\mathbb{R}^{(m,n)}\right)$ and $\omega\in\mathbb{Z}^{(m,n)}$, using Formulas (2.4), (3.10) or (4.6), we have

$$\begin{aligned}
& \left[\pi_{\mathcal{M}}\big((0,s;0)T_*\big)\,\pi_{\mathcal{M}}\big((\lambda,\mu;\kappa)(\tau,\phi)\big)f\right](\omega) \\
= \;& \left[\mathscr{W}_{\mathcal{M}}(0,s;0)R_{\mathcal{M}}(T_*)\,\pi_{\mathcal{M}}\big((\lambda,\mu;\kappa)(\tau,\phi)\big)f\right](\omega) \\
= \;& e^{2\pi i\,\sigma(\mathcal{M}\omega\,{}^ts)}\cdot e^{2\,\|\omega\|_{\mathcal{M}}^2\pi\,i}\,\left[R_{\mathcal{M}}(i,0)\,\pi_{\mathcal{M}}\big((\lambda,\mu;\kappa)(\tau,\phi)\big)f\right](\omega) \\
= \;& \left[\pi_{\mathcal{M}}\big((\lambda,\mu;\kappa)(\tau,\phi)\big)f\right](\omega).
\end{aligned}$$

Here we used the facts that

$$e^{2\pi i\,\sigma(\mathcal{M}\omega\,{}^ts)}=1,\quad e^{2\,\|\omega\|_{\mathcal{M}}^2\pi\,i}=1\quad\text{and}\quad R_{\mathcal{M}}(i,0)f=f\ \text{(cf. (4.7))}.$$

Therefore for $f \in \mathscr{S}\left(\mathbb{R}^{(m,n)}\right)$,

$$\begin{aligned}
& \Theta_f^{[\mathcal{M}]}(\tau+2,\phi\,;\lambda, s-2\,\lambda+\mu, \kappa - s\,{}^t\lambda) \\
= & \sum_{\omega\in\mathbb{Z}^{(m,n)}} \left[\pi_{\mathcal{M}}\big((\lambda, s-2\,\lambda+\mu, \kappa-s\,{}^t\lambda)(\tau+2,\phi)\big)f\right](\omega) \\
= & \sum_{\omega\in\mathbb{Z}^{(m,n)}} \left[\pi_{\mathcal{M}}\big((0,s;0)T_*\big)\,\pi_{\mathcal{M}}\big((\lambda,\mu;\kappa)(\tau,\phi)\big)f\right](\omega) \\
= & \sum_{\omega\in\mathbb{Z}^{(m,n)}} \left[\pi_{\mathcal{M}}\big((\lambda,\mu;\kappa)(\tau,\phi)\big)f\right](\omega) \\
= & \Theta_f^{[\mathcal{M}]}(\tau,\phi\,;\lambda,\mu,\kappa).
\end{aligned}$$

This completes the proof. $\square$

Theorem 4.3 (Jacobi 3)**.** *Let $\mathcal{M}=(\mathcal{M}_{kl})$ be a positive definite symmetric integral $m\times m$ matrix and let $(\lambda_0,\mu_0;\kappa_0)\in H_{\mathbb{Z}}^{(m,n)}$ be an integral element of $H_{\mathbb{R}}^{(n,m)}$. Then we have*

$$\begin{aligned}
& \Theta_f^{[\mathcal{M}]}(\tau,\phi\,;\lambda+\lambda_0, \mu+\mu_0, \kappa+\kappa_0+\lambda_0\,{}^t\mu-\mu_0\,{}^t\lambda) \\
= & \; e^{\pi\, i\,\sigma(\mathcal{M}(\kappa_0+\mu_0\,{}^t\lambda_0))}\Theta_f^{[\mathcal{M}]}(\tau,\phi\,;\lambda,\mu,\kappa)
\end{aligned}$$

for all $(\tau,\phi)\in SL(2,\mathbb{R})$ and $(\lambda,\mu;\kappa)\in H_{\mathbb{R}}^{(n,m)}$.

Proof. For any $f \in \mathscr{S}\left(\mathbb{R}^{(m,n)}\right)$, we have

$$\begin{aligned}
& \sum_{\omega\in\mathbb{Z}^{(m,n)}} \left[\mathscr{W}_{\mathcal{M}}(\lambda_0,\mu_0;\kappa_0)\pi_{\mathcal{M}}\big((\lambda,\mu\,;\kappa)(\tau,\phi)\big)f\right](\omega) \\
= & \sum_{\omega\in\mathbb{Z}^{(m,n)}} \left[\mathscr{W}_{\mathcal{M}}(\lambda_0,\mu_0;\kappa_0)\mathscr{W}_{\mathcal{M}}(\lambda,\mu\,;\kappa)R_{\mathcal{M}}(\tau,\phi)f\right](\omega) \\
= & \sum_{\omega\in\mathbb{Z}^{(m,n)}} \left[\mathscr{W}_{\mathcal{M}}(\lambda_0+\lambda,\mu_0+\mu;\kappa_0+\kappa+\lambda_0\,{}^t\mu-\mu_0\,{}^t\lambda))R_{\mathcal{M}}(\tau,\phi)f\right](\omega) \\
= & \sum_{\omega\in\mathbb{Z}^{(m,n)}} \left[\pi_{\mathcal{M}}\big((\lambda_0+\lambda,\mu_0+\mu;\kappa_0+\kappa+\lambda_0\,{}^t\mu-\mu_0\,{}^t\lambda)(\tau,\phi)\big)f\right](\omega) \\
= & \Theta_f^{[\mathcal{M}]}(\tau,\phi\,;\lambda+\lambda_0,\mu+\mu_0,\kappa+\kappa_0+\lambda_0\,{}^t\mu-\mu_0\,{}^t\lambda).
\end{aligned}$$

On the other hand, for any $f \in \mathscr{S}\left(\mathbb{R}^{(m,n)}\right)$, we have

$$\begin{aligned}
& \sum_{\omega\in\mathbb{Z}^{(m,n)}} \left[\mathscr{W}_{\mathcal{M}}(\lambda_0,\mu_0;\kappa_0)\pi_{\mathcal{M}}\big((\lambda,\mu\,;\kappa)(\tau,\phi)\big)f\right](\omega) \\
= & \sum_{\omega\in\mathbb{Z}^{(m,n)}} e^{\pi i\sigma\{\mathcal{M}(\kappa_0+\mu_0\,{}^t\lambda_0+2\,\omega\,{}^t\mu_0)\}}\left[\pi_{\mathcal{M}}(\tau,\phi\,;\lambda,\mu,\kappa)f\right](\omega+\lambda_0) \\
= & \; e^{\pi i\sigma\{\mathcal{M}(\kappa_0+\mu_0\,{}^t\lambda_0\}}\sum_{\omega\in\mathbb{Z}^{(m,n)}} \left[\pi_{\mathcal{M}}(\tau,\phi\,;\lambda,\mu,\kappa)f\right](\omega+\lambda_0) \quad (\because\ \mu_0 \text{ is integral}) \\
= & \; e^{\pi i\sigma\{\mathcal{M}(\kappa_0+\mu_0\,{}^t\lambda_0\}}\sum_{\omega\in\mathbb{Z}^{(m,n)}} \left[\pi_{\mathcal{M}}(\tau,\phi\,;\lambda,\mu,\kappa)f\right](\omega) \quad (\because\ \lambda_0 \text{ is integral}) \\
= & \; e^{\pi i\sigma\{\mathcal{M}(\kappa_0+\mu_0\,{}^t\lambda_0\}}\,\Theta_f^{[\mathcal{M}]}(\tau,\phi\,;\lambda,\mu,\kappa).
\end{aligned}$$

Finally we obtain the desired result. □

We put $V(m,n) = \mathbb{R}^{(m,n)}\times\mathbb{R}^{(m,n)}$. Let

$$G^{(m,n)} := SL(2,\mathbb{R}) \ltimes V(m,n)$$

be the group with the following multiplication law

$$(g_1,(\lambda_1,\mu_1))\cdot(g_2,(\lambda_2,\mu_2)) = (g_1g_2,(\lambda_1,\mu_1)g_2+(\lambda_2,\mu_2)), \tag{4.15}$$

where $g_1,g_2\in SL(2,\mathbb{R})$ and $\lambda_1,\lambda_2,\mu_1,\mu_2\in\mathbb{R}^{(m,n)}$.

We define

$$\Gamma^{(m,n)} := SL(2,\mathbb{Z}) \ltimes H_{\mathbb{Z}}^{(n,m)}.$$

Then $\Gamma^{(m,n)}$ acts on $G^{(m,n)}$ naturally through the multiplication law (4.15).

Lemma 4.3. *$\Gamma^{(m,n)}$ is generated by the elements*

$$(S,(0,0)),\quad (T_\flat,(0,s))\quad \textit{and}\quad (I_2,(\lambda_0,\mu_0)),$$

where

$$S=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\quad T_\flat=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\quad \textit{and } s,\lambda_0,\mu_0\in\mathbb{Z}^{(m,n)}.$$

Proof. Since $SL(2,\mathbb{Z})$ is generated by S and $T_\flat$, we get the desired result. □

We define

$$\begin{aligned}
& \Theta_f^{[\mathcal{M}]}(\tau,\phi;\lambda,\mu) \\
= & \; v^{\frac{mn}{4}}\sum_{\omega\in\mathbb{Z}^{(m,n)}} e^{\pi\,i\,\left\{u\|\omega+\lambda\|^2_{\mathcal{M}}+2(\omega,\mu)_{\mathcal{M}}\right\}}\left[R_{\mathcal{M}}(i,\phi)f\right]\Big(v^{1/2}(\omega+\lambda)\Big).
\end{aligned}$$

Theorem 4.4. *Let $\Gamma^{(m,n)}_{[2]}$ be the subgroup of $\Gamma^{(m,n)}$ generated by the elements*

$$(S,(0,0)),\quad (T_*,(0,s))\quad \textit{and}\quad (I_2,(\lambda_0,\mu_0)),$$

where

$$T_*=\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}\quad \textit{and } s,\lambda_0,\mu_0\in\mathbb{Z}^{(m,n)}.$$

Let $\mathcal{M} = (\mathcal{M}_{kl})$ *be a positive definite symmetric unimodular integral* $m \times m$ *matrix such that* $\mathcal{M}\mathbb{Z}^{(m,n)} = \mathbb{Z}^{(m,n)}$. *Then for* $f, g \in \mathscr{S}\left(\mathbb{R}^{(m,n)}\right)$, *the function*

$$\Theta_f^{[\mathcal{M}]}(\tau,\phi;\lambda,\mu)\,\overline{\Theta_g^{[\mathcal{M}]}(\tau,\phi;\lambda,\mu)}$$

is invariant under the action of $\Gamma_{[2]}^{(m,n)}$ *on* $G^{(m,n)}$.

Proof. The proof follows directly from Theorem 4.1 (Jacobi 1), Theorem 4.2 (Jacobi 2) and Theorem 4.3 (Jacobi 3) because the left actions of the generators of $\Gamma_{[2]}^{(m,n)}$ are given by

$$((\tau,\phi),(\lambda,\mu)) \longmapsto \left(\left(-\frac{1}{\tau}, \phi + \arg\tau\right), (-\mu,\lambda)\right),$$
$$((\tau,\phi),(\lambda,\mu)) \longmapsto ((\tau+2,\phi),(\lambda, s-2\,\lambda+\mu))$$

and

$$((\tau,\phi),(\lambda,\mu)) \longmapsto ((\tau,\phi),(\lambda+\lambda_0,\mu+\mu_0)).$$

□

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