Geometry on the Siegel-Jacobi Space

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Preface

The Siegel upper half plane (briefly the Siegel space)

$$\mathbb{H}_n := \{\Omega \in \mathbb{C}^{(n,n)} \mid \Omega = {}^t\Omega,\ \operatorname{Im}\Omega > 0\}^1.$$

is one of the classical domains due to Élie Cartan (1869 − 1951) and plays an important role in various mathematical branches, for instance, number theory, algebraic geometry, differential geometry, mathematical physics etc. The Siegel space is a high dimensional version of the Poincaré upper half plane. The Siegel space is an Einstein-Kähler Hermitian symmetric manifold. Thus the Siegel space provides the beautiful and profound theory. Many geometrical and number-theoretical properties on the Siegel space have been investigated by many outstanding mathematicians, e.g., Carl Siegel (1896 − 1981), Hans Maaß (1911 − 1992), Ichiro Satake (1927 − 2014), Goro Shimura (1930 − 2018), David Mumford (1937–) and so on.

The Siegel-Jacobi space (briefly the S-J space) $\mathbb{H}_{n,m} := \mathbb{H}_n \times \mathbb{C}^{(m,n)}$ is also an important Kähler homogeneous space which is *not* Einstein and non-symmetric. The arithmetic quotient of the S-J space appears naturally in the boundary of a toroidal compactification of the Siegel modular variety. Thus the study of the S-J space should be carried out in order to understand the boundary behaviour of a toroidal compactification of the Siegel modular variety. The study of an arithmetic quotient of the S-J space had been started by Michio Kuga (1928 − 1990) in the special context of Kuga fibre varieties from the early 1960s. Kuga varieties are fiber varieties over symmetric spaces whose fibers are abelian varieties and have played an important role in the theory of Shimura varieties and number theory. The S-J space $\mathbb{H}_{n,m}$ may be considered as a combination of the hyperbolic space $\mathbb{H}_n$ and the Euclidean space $\mathbb{C}^{(m,n)}$.

The theory of Jacobi forms and the representation theory of the Jacobi group G^J (the semidirect product of the symplectic group and the Heisen-

[1] $\mathbb{C}^{(k,l)}$, ${}^t\Omega$ and $\operatorname{Im}\Omega$ denote the space of all $k \times l$ complex matrices, the transpose of Ω and the imaginary part of Ω respectively.

berg group) have been extensively developed after the famous book of M. Eichler and D. Zagier, *The Theory of Jacobi Forms* was published in 1985. Recently Jacobi forms are usefully applied in number theory, geometry, representation theory and mathematical physics. The author started to study Jacobi forms of high degree in the late 1980. At that time I realized that the S-J space had not been studied well enough and that the S-J space provides new and profound problems in the various area of number theory, algebraic geometry, differential geometry, invariant theory, complex function theory, representation theory and mathematical physics.

In this book, the author gives some basic geometric results on the S-J space and propose important and interesting open problems to be investigated and solved in the near future. In order to attack and answer the open problems, one needs complicated and tedious computation techniques. The computation transcend that in the Siegel space. The author was very happy when he discovered the Laplace operators of invariant metrics on the S-J space around the early 2000s after several mistakes in the complicated computations. The author tried to solve the problems about *invariant theory* coming naturally from the study of invariant differential operators on the S-J space under the *natural action* of the Jacobi group on the S-J space. Unfortunately the author did not succeed in solving these open problems even though he learned a lot about invariant theory. He proposed these invariant-theoretical open problems in the already-published articles and at the conferences and colloquium talks. Some other young mathematicians were very interested in these open problems and thus tried to solve them. But they also did not succeed in solving these problems until now.

The S-J space is rich both geometric natures and algebraic natures. It provides an important common ground for various branches of mathematics and physics, not only for number theory, algebraic geometry, differential geometry, arithmetic geometry, but also representation theory, mathematical physics. The author personally thinks that the S-J space is a mathematical subject which transcend time and space. He believes that the profound gems will appear brightly and lightfully in the deep study of the S-J space.

Finally the author hopes that this book will be a good reference to the young mathematicians who are interested in the geometry and arithmetic of the S-J space, and also to physicists for the study related to the S-J space.

Introduction

For a given fixed positive integer n, we let

$$\mathbb{H}_n = \{\,\Omega \in \mathbb{C}^{(n,n)} \mid \Omega = {}^t\Omega, \quad \operatorname{Im}\Omega > 0\,\}$$

be the Siegel upper half plane of degree n and let

$$Sp(2n,\mathbb{R}) = \{M \in \mathbb{R}^{(2n,2n)} \mid {}^tM\, J_n M = J_n\ \}$$

be the symplectic group of degree n, where $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F for two positive integers k and l, ${}^tM$ denotes the transposed matrix of a matrix M and

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Then $Sp(2n,\mathbb{R})$ acts on $\mathbb{H}_n$ naturally and transitively by

$$(\mathfrak{S}) \qquad M\langle\Omega\rangle = (A\Omega + B)(C\Omega + D)^{-1},$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2n,\mathbb{R})$ and $\Omega \in \mathbb{H}_n$. The symmetric space $\mathbb{H}_n = Sp(2n,\mathbb{R})/U(n)$ is a non-Euclidean hyperbolic space and also a Kähler-Einstein manifold. Let

$$\Gamma_n = Sp(2n,\mathbb{Z}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2n,\mathbb{R}) \,\middle|\, A, B, C, D \text{ integral} \right\}$$

be the Siegel modular group of degree n. This group acts on $\mathbb{H}_n$ properly discontinuously. C. L. Siegel investigated the geometry of $\mathbb{H}_n$ and automorphic forms on $\mathbb{H}_n$ systematically. Siegel [163] found a fundamental domain $\mathcal{F}_n$ for $\Gamma_n\backslash\mathbb{H}_n$ and described it explicitly. Moreover he calculated the volume of $\mathcal{F}_n$. We also refer to [89], [118], [163] for some details on $\mathcal{F}_n$. The Siegel modular variety

$$\mathcal{A}_n := \Gamma_n\backslash\mathbb{H}_n$$

is one of the important arithmetic varieties in the sense that it is regarded as the moduli of principally polarized abelian varieties of dimension n. Suggested by Siegel, I. Satake [149] found a canonical compactification, now called the Satake compactification of $\mathcal{A}_n$. Thereafter W. Baily [3] proved that the Satake compactification of $\mathcal{A}_n$ is a normal projective variety. This work was generalized to bounded symmetric domains by W. Baily and A. Borel [5, 6] around the 1960s. Some years later a theory of smooth compactification of bounded symmetric domains was developed by Mumford school [2]. G. Faltings and C.-L. Chai [45] investigated the moduli of abelian varieties over the integers and could give the analogue of the Eichler-Shimura theorem that expresses Siegel modular forms in terms of the cohomology of local systems on $\mathcal{A}_n$. I want to emphasize that Siegel modular forms play an important role in the theory of the arithmetic and the geometry of the Siegel modular variety $\mathcal{A}_n$.

The Siegel-Jacobi space

$$\mathbb{H}_{n,m} := \mathbb{H}_n \times \mathbb{C}^{(m,n)}$$

is a very important non-symmetric homogeneous Käher manifold geometrically and arithmetically. Here $\mathbb{C}^{(m,n)}$ is the Euclidean space consisting of all $m \times n$ complex matrices. The Siegel-Jacobi space $\mathbb{H}_{n,m}$ is obtained as a homogeneous manifold as follows.

For two positive integers m and n, we consider the Heisenberg group

$$H_{\mathbb{R}}^{(n,m)} = \{\, (\lambda, \mu, \kappa) \mid \lambda, \mu \in \mathbb{R}^{(m,n)},\ \kappa \in \mathbb{R}^{(m,m)},\ \kappa + \mu\, {}^t\lambda \text{ symmetric } \}$$

endowed with the following multiplication law

$$(\lambda, \mu, \kappa) \circ (\lambda', \mu', \kappa') = (\lambda + \lambda', \mu + \mu', \kappa + \kappa' + \lambda\, {}^t\mu' - \mu\, {}^t\lambda')$$

with $(\lambda, \mu, \kappa), (\lambda', \mu', \kappa') \in H_{\mathbb{R}}^{(n,m)}$. We define the Jacobi group G^J of degree n and index m that is the semidirect product of $Sp(2n, \mathbb{R})$ and $H_{\mathbb{R}}^{(n,m)}$

$$G^J = Sp(2n, \mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)}$$

endowed with the following multiplication law

$$\begin{aligned} & (M, (\lambda, \mu; \kappa)) \cdot (M', (\lambda', \mu'; \kappa')) \\ = \;& (MM', (\tilde{\lambda} + \lambda', \tilde{\mu} + \mu'; \kappa + \kappa' + \tilde{\lambda}\, {}^t\mu' - \tilde{\mu}\, {}^t\lambda')) \end{aligned}$$

with $M, M' \in Sp(2n, \mathbb{R}), (\lambda, \mu, \kappa), (\lambda', \mu', \kappa') \in H_{\mathbb{R}}^{(n,m)}$ and $(\tilde{\lambda}, \tilde{\mu}) = (\lambda, \mu)M'$. Then G^J acts on $\mathbb{H}_{n,m}$ *naturally* by

$$(\mathfrak{S}\text{–}\mathfrak{J}) \quad (M, (\lambda, \mu, \kappa)) \cdot (\Omega, Z) = \Big(M\langle \Omega \rangle, (Z + \lambda\Omega + \mu)(C\Omega + D)^{-1} \Big),$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2n,\mathbb{R}),\ (\lambda,\mu,\kappa) \in H_{\mathbb{R}}^{(n,m)}$ and $(\Omega, Z) \in \mathbb{H}_{n,m}$. We see easily that the action ($\mathfrak{S}$–$\mathfrak{J}$) is transitive. We note that the Jacobi group G^J is *not* a reductive Lie group. Then the Siegel-Jacobi space $\mathbb{H}_{n,m}$ is biholomorphic to the non-symmetric homogeneous Kähler manifold

$$G^J/K_{n,m} \cong \mathbb{H}_{n,m}, \qquad K_{n,m} := U(n) \times \mathrm{Symm}(m,\mathbb{R}),$$

where $\mathrm{Symm}(m,\mathbb{R})$ is the space of all $m \times m$ real symmetric matrices. Let Γ be a subgroup of Γ_n of finite index which acts on $\mathbb{H}_n$. Then

$$\Gamma^J := \Gamma \ltimes H_{\mathbb{Z}}^{(n,m)}$$

acts on $\mathbb{H}_{n,m}$ properly discontinuously. Then the Siegel-Jacobi modular variety

$$\mathcal{A}_{n,m,\Gamma} := \Gamma^J \backslash \mathbb{H}_{n,m}$$

may be considered as the universal family of abelian varieties, that is, roughly the fibre bundle over the Siegel modular variety $\mathcal{A}_{n,\Gamma} := \Gamma\backslash\mathbb{H}_n$ with fibres as abelian varieties.

As you see, $\mathbb{H}_n$, $\mathbb{C}^{(m,n)}$, $Sp(2n,\mathbb{R})$, $H_{\mathbb{R}}^{(n,m)}$, $\mathbb{H}_{n,m}$ and $\mathcal{A}_{n,m,\Gamma}$ are important geometrical objects. I have investigated the Siegel-Jacobi space geometrically and arithmetically past thirty years. In this book I develop the geometrical theory of the Siegel-Jacobi space. I describe the geometric results I obtained before and add new geometric results in this book. And I present some open geometric problems to be solved in the future.

The book is organized as follows. In Chapter 1, we review the basic properties of the symplectic group $Sp(2n,\mathbb{R})$. In Chapter 2, we review the well-known geometric results of the Siegel upper half plane (simply *the Siegel space*) obtained by Siegel, Maaß and others. We realize the Siegel space as the classifying space of Hodege structures of weight one. We also realize the Siegel space as the period matrix for a compact Riemann surface. In Chapter 3, we deal with the subject of differential operators on the Siegel space $\mathbb{H}_n$ invariant under the *natural* action ($\mathfrak{S}$). We review the important results on invariant differential operators on $\mathbb{H}_n$ obtained by Maaß, Harish-Chandra and Shimura. In Chapter 4, we review the Satake compactification and the toroidal compactifications of the Siegel modular variety. In Chapter 5, we review the basic properties of the Jacobi group. In Chapter 6, we investigate the differential geometric properties of the Siegel-Jacobi space. We deal with invariant metrics, Laplacians, the partial Cayley transform, a fundamental domain, curvatures and so on. In Chapter 7, we construct the canonical automorphic factor for the Siegel-Jacobi space in a geometrical method. This automorphic factor is used to define the notion of Jacobi forms in Chapter 9. In Chapter 8, we discuss the theory of differential operators on the Siegel-Jacobi space $\mathbb{H}_{n,m}$ invariant under the *natural* action ($\mathfrak{S}$–$\mathfrak{J}$). This theory is still not well developed yet. This theory should be investigated in detail sooner or later. We present open problems. In Chapter 9, we define the concept of Jacobi forms using the canonical automorphic factor constructed in Chapter 7. We briefly review the basic properties and well-known results about Jacobi forms. In Chapter 10, we introduce the notions of Maaß-Jacobi forms in several ways which are useful to develop the theory of harmonic analysis on the Siegel-Jacobi space. In Chapter 11, we deal with the toroidal compactification of the Siegel-Jacobi modular variety briefly and roughly. It is very complicated and difficult to investigate this topic. We explain how Jacobi forms are used to study the Siegel-Jacobi modular variety. We show that Jacobi forms can be characterized as sections of the automorphic vector bundle over the Siegel-Jacobi modular variety. We discuss how Jacobi forms appear in the study of the boundary behaviour of the Satake compactification of the Siegel modular variety. In Chapter 12, using the theory of compactifications of the Siegel-Jacobi modular variety studied in Chapter 11, we develop the theory of the stability of the Siegel-Jacobi modular varieties and the theory of the stability of Jacobi forms. In Chapter 13, we propose geometric open problems that should be solved and studied in the future. In the final Chapter, we make some brief remarks on the geometry of the Minkowski-Euclid space, the invariant theory and the theory of compactifications of locally symmetric spaces. We also provides some open problems arising from the study of invariant differential operators on the Minkowski-Euclid space and the Siegel-Jacobi space.

[Chapter 12]

[Chapter 13]

[Chapter 14]