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HARMONIC ANALYSIS ON THE QUOTIENT SPACES OF HEISENBERG GROUPS

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A certain nilpotent Lie group plays an important role in the study of the foundations of quantum mechanics ([Wey]) and of the theory of theta series (see [C], [I] and [Wei]). This work shows how theta series are applied to decompose the natural unitary representation of a Heisenberg group.

For any positive integers g and h, we consider the Heisenberg group

$$H_R^{(g,h)} := \{[(\lambda, \mu), \kappa] \mid \lambda, \mu \in R^{(h,g)},\ \kappa \in R^{(h,h)},\ \kappa + \mu\, {}^t\lambda \text{ symmetric}\}$$

endowed with the following multiplication law

$$[(\lambda, \mu), \kappa] \circ [(\lambda', \mu'), \kappa'] = [(\lambda + \lambda', \mu + \mu'), \kappa + \kappa' + \lambda\, {}^t\mu' - \mu\, {}^t\lambda'] .$$

The mapping

$$H_R^{(g,h)} \ni [(\lambda, \mu), \kappa] \longrightarrow \begin{pmatrix} E_g & 0 & 0 & {}^t\mu \\ \lambda & E_h & \mu & \kappa \\ 0 & 0 & E_g & -{}^t\lambda \\ 0 & 0 & 0 & E_h \end{pmatrix}$$

defines an embedding of $H_R^{(g,h)}$ into the symplectic group $Sp(g+h, R)$. We refer to [Z] for the motivation of the study of this Heisenberg group $H_R^{(g,h)}$. $H_Z^{(g,h)}$ denotes the discrete subgroup of $H_R^{(g,h)}$ consisting of integral elements, and $L^2(H_Z^{(g,h)} \backslash H_R^{(g,h)})$ is the L^2-space of the quotient space $H_Z^{(g,h)} \backslash H_R^{(g,h)}$ with respect to the invariant measure

$$d\lambda_{11} \cdots d\lambda_{h,g-1} d\lambda_{hg} d\mu_{11} \cdots d\mu_{h,g-1} d\mu_{hg} d\kappa_{11} d\kappa_{12} \cdots d\kappa_{h-1,h} d\kappa_{hh}.$$

We have the natural unitary representation ρ on $L^2(H_Z^{(g,h)} \backslash H_R^{(g,h)})$ given by

$$\rho([(\lambda', \mu'), \kappa'])\phi([(\lambda, \mu), \kappa]) = \phi([(\lambda, \mu), \kappa] \circ [(\lambda', \mu'), \kappa']).$$

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The Stone-von Neumann theorem says that an irreducible representation ρ of $H_R^{(g,h)}$ is characterized by a real symmetric matrix $c \in R^{(h,h)}$ $(c \neq 0)$ such that

$$\rho_c([(0, 0), \kappa]) = \exp\{\pi i \sigma(c\kappa)\} I, \qquad \kappa = {}^t\kappa \in R^{(h,h)},$$

where I denotes the identity mapping of the representation space. If $c = 0$, then it is characterized by a pair $(k, m) \in R^{(h,g)} \times R^{(h,g)}$ such that

$$\rho_{k,m}([(\lambda, \mu), \kappa]) = \exp\{2\pi i \sigma(k\,{}^t\lambda + m\,{}^t\mu)\} I .$$

But only the irreducible representations $\rho_{\mathcal{M}}$ with $\mathcal{M} = {}^t\mathcal{M}$ even integral and $\rho_{k,m}$ $(k, m \in Z^{(h,g)})$ could occur in the right regular representation ρ in $L^2(H_Z^{(g,h)} \backslash H_R^{(g,h)})$.

In this article, we decompose the right regular representation ρ. The real analytic functions defined in (1.5) play an important role in decomposing the right regular representation ρ.

NOTATIONS. We denote Z, R and C the ring of integers, the field of real numbers and the field of complex numbers respectively. $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F. E_g denotes the identity matrix of degree g. $\sigma(A)$ denotes the trace of a square matrix A.

$$\begin{aligned}
Z_{\geq 0}^{(h,g)} &= \{J = (J_{kl}) \in Z^{(h,g)} \mid J_{kl} \geq 0 \text{ for all } k, l\}, \\
|J| &= \sum_{k,l} J_{kl}, \\
J \pm \varepsilon_{kl} &= (J_{11}, \cdots, J_{kl} \pm 1, \cdots, J_{hg}), \\
(\lambda + N + A)^J &= (\lambda_{11} + N_{11} + A_{11})^{J_{11}} \cdots (\lambda_{hg} + N_{hg} + A_{hg})^{J_{hg}} .
\end{aligned}$$

§ 1. Theta series

Let H_g be the Siegel upper half plane of degree g. We fix an element $\Omega \in H_g$ once and for all. Let $\mathcal{M}$ be a positive definite, symmetric even integral matrix of degree h. A holomorphic function $f\colon C^{(h,g)} \to C$ satisfying the functional equation

$$f(W + \lambda\Omega + \mu) = \exp\{-\pi i \sigma(\mathcal{M}(\lambda\Omega\,{}^t\lambda + 2\lambda\,{}^tW))\} f(W) \tag{1.1}$$

for all $\lambda, \mu \in Z^{(h,g)}$ is called a *theta series* of *level* $\mathcal{M}$ with respect to Ω. The set $T_{\mathcal{M}}(\Omega)$ of all theta series of level $\mathcal{M}$ with respect to Ω is a vector space of dimension $(\det \mathcal{M})^g$ with a basis consisting of theta series

(1.2)

$$\vartheta^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega, W) := \sum_{N \in Z^{(h,g)}} \exp\{\pi i \sigma\{\mathcal{M}((N+A)\Omega\,{}^t(N+A) + 2W\,{}^t(N+A))\}\},$$

where A runs over a complete system of representatives of the cosets $\mathcal{M}^{-1}Z^{(h,g)}/Z^{(h,g)}$.

DEFINITION 1.1. A function φ: $C^{(h,g)} \times C^{(h,g)} \to C$ is called an *auxiliary theta series* of *level* $\mathcal{M}$ with respect to Ω if it satisfies the following conditions (i) and (ii):

(i) $\varphi(U, W)$ is a polynomial in W whose coefficients are entire functions,

(ii) $\varphi(U + \lambda, W + \lambda\Omega + \mu) = \exp\{-\pi i(\mathcal{M}(\lambda\Omega\,{}^t\lambda + 2\lambda\,{}^tW))\}\varphi(U, W)$ for all $(\lambda, \mu) \in Z^{(h,g)} \times Z^{(h,g)}$.

The space $\Theta_\Omega^{(\mathcal{M})}$ of all auxiliary theta series of level $\mathcal{M}$ with respect to Ω has a basis consisting of the following functions:

$$(1.3)\quad \vartheta_J^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega \mid \lambda, \mu + \lambda\Omega) := \sum_{N \in Z^{(h,g)}} (\lambda + N + A)^J \times \exp\{\pi i \sigma(\mathcal{M}((N+A)\Omega\,{}^t(N+A) + (\mu + \lambda\Omega)\,{}^t(N+A)))\}.$$

where A (resp. J) runs over the cosets $\mathcal{M}^{-1}Z^{(h,g)}/Z^{(h,g)}$ (resp. $Z_{\geq 0}^{(h,g)}$).

DEFINITION 1.2. A real analytic function φ: $R^{(h,g)} \times R^{(h,g)} \to C$ is called a *mixed theta series* of *level* $\mathcal{M}$ with respect to Ω if φ satisfies the following conditions (1) and (2):

(1) $\varphi(\lambda, \mu)$ is a polynomial in λ whose coefficients are entire functions in complex variables $Z = \mu + \lambda\Omega$;

(2) $\varphi(\lambda + \tilde{\lambda}, \mu + \tilde{\mu}) = \exp\{-\pi i \sigma(\mathcal{M}(\tilde{\lambda}\Omega\,{}^t\tilde{\lambda} + 2(\mu + \lambda\Omega)\,{}^t\tilde{\lambda}))\}\varphi(\lambda, \mu)$ for all $(\tilde{\lambda}, \tilde{\mu}) \in Z^{(h,g)} \times Z^{(h,g)}$.

If $A \in \mathcal{M}^{-1}Z^{(h,g)}/Z^{(h,g)}$ and $J \in Z_{\geq 0}^{(h,g)}$,

$$(1.4)\quad \vartheta_J^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega \mid \lambda, \mu + \lambda\Omega) := \sum_{N \in Z^{(h,g)}} (\lambda + N + A)^J \times \exp\{\pi i \sigma(\mathcal{M}((N+A)\Omega\,{}^t(N+A) + 2(\mu + \lambda\Omega)\,{}^t(N+A)))\}$$

is a mixed theta series of level $\mathcal{M}$.

Now for a positive definite symmetric even integral matrix $\mathcal{M}$ of degree h, we define a function on $H_R^{(g,h)}$.

$$(1.5)\quad \Phi_J^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega \mid [(\lambda, \mu), \kappa]) := \exp\{\pi i \sigma(\mathcal{M}(\kappa - \lambda\,{}^t\mu))\} \sum_{N \in Z^{(h,g)}} (\lambda + N + A)^J \times \exp\{\pi i \sigma(\mathcal{M}((\lambda + N + A)\Omega\,{}^t(\lambda + N + A) + 2(\lambda + N + A)\,{}^t\mu))\},$$

where $A \in \mathscr{M}^{-1}Z^{(h,g)}/Z^{(h,g)}$.

PROPOSITION 1.3.

$$(1.6)\quad \Phi_J^{(\mathscr{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa]) = \exp\{2\pi i\sigma(\mathscr{M}\mu\,{}^tA)\}\Phi_J^{(\mathscr{M})}\begin{bmatrix} 0 \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa]\circ[(A,0),0]).$$

$$(1.7)\quad \Phi_J^{(\mathscr{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\tilde\lambda,\tilde\mu),\tilde\kappa]\circ[(\lambda,\mu),\kappa]) = \Phi_J^{(\mathscr{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa]).$$

$$([(\tilde\lambda,\tilde\mu),\tilde\kappa]\in H_Z^{(h,g)},\ [(\lambda,\mu),\kappa]\in H_R^{(h,g)},\ A\in\mathscr{M}^{-1}Z^{(h,g)}/Z^{(h,g)}).$$

Proof.

$$\begin{aligned}
&\Phi_J^{(\mathscr{M})}\begin{bmatrix} 0 \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda+A,\mu),\kappa-\mu\,{}^tA])\\
&= \exp\{\pi i\sigma(\mathscr{M}(\kappa-\mu\,{}^tA-(\lambda+A)\,{}^t\mu))\}\sum_{N\in Z^{(h,g)}}(\lambda+A+N)^J\\
&\quad\times\exp\{\pi i\sigma(\mathscr{M}((\lambda+A+N)\Omega\,{}^t(\lambda+N+A)+2(\lambda+N+A)\,{}^t\mu))\}\\
&= \exp\{-2\pi i\sigma(\mathscr{M}\mu\,{}^tA)\}\Phi_J^{(\mathscr{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa]).
\end{aligned}$$

On the other hand, if $[(\tilde\lambda,\tilde\mu),\tilde\kappa]\in H_Z^{(h,g)}$,

$$\begin{aligned}
&\Phi_J^{(\mathscr{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\tilde\lambda,\tilde\mu),\tilde\kappa]\circ[(\lambda,\mu),\kappa])\\
&= \exp\{\pi i\sigma(\mathscr{M}(\tilde\kappa+\kappa+\tilde\lambda\,{}^t\mu-\tilde\mu\,{}^t\lambda-(\tilde\lambda+\lambda)\,{}^t(\tilde\mu+\mu)))\}\sum_{N\in Z^{(h,g)}}(\tilde\lambda+\lambda+N+A)^J\\
&\quad\times\exp\{\pi i\sigma((\tilde\lambda+\lambda+N+A)\Omega\,{}^t(\tilde\lambda+\lambda+N+A)+2(\tilde\lambda+\lambda+N+A)\,{}^t(\tilde\mu+\mu))\}\\
&= \Phi_J^{(\mathscr{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa]).
\end{aligned}$$

Here in the last equality we used the facts that $\sigma(\mathscr{M}(\tilde\kappa-{}^t\tilde\lambda\tilde\kappa))\in 2Z$ and $\sigma(\mathscr{M}A\,{}^t\tilde\mu)\in Z$. q.e.d.

Remark. Proposition 1.3 implies that $\Phi_J^{(\mathscr{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa])\,(J\in Z_{\geq 0}^{(h,g)})$ are real analytic functions on the quotient space $H_Z^{(g,h)}\backslash H_R^{(g,h)}$.

The following matrices

$$X_{kl}^0 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & E_{kl}^0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\quad 1\leq k\leq l\leq h,$$

$$\hat{X}_{ij} := \begin{pmatrix} 0 & 0 & 0 & {}^tE_{ij} \\ 0 & 0 & E_{ij} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad 1 \le i \le h, \quad 1 \le j \le g,$$

$$X_{ij} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ E_{ij} & 0 & 0 & 0 \\ 0 & 0 & 0 & -{}^tE_{ij} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad 1 \le i \le h, \quad 1 \le j \le g$$

form a basis of the Lie algebra $\mathscr{H}_R^{(g,h)}$ of the Heisenberg group $H_R^{(g,h)}$. Here E_{kl}^0 $(k \neq l)$ and $h \times h$ symmetric matrix with entry 1/2 where the k-th (or l-th) row and the l-th (or k-th) column meet, all other entries 0, E_{kk}^0 is an $h \times h$ diagonal matrice with the k-th diagonal entry 1 and all other entries 0 and E_{ij} is an $h \times g$ matrix with entry 1 where the i-th row and the j-th column meet, all other entries 0. By an easy calculation, we see that the following vector fields

$$D_{kl}^0 = \frac{\partial}{\partial \kappa_{kl}}, \qquad 1 \le k \le l \le h,$$

$$D_{mp} = \frac{\partial}{\partial \lambda_{mp}} - \left(\sum_{k=1}^{m} \mu_{kp} \frac{\partial}{\partial \kappa_{km}} + \sum_{k=m+1}^{h} \mu_{kp} \frac{\partial}{\partial \kappa_{mk}} \right),$$

$$\hat{D}_{mp} = \frac{\partial}{\partial \mu_{mp}} + \left(\sum_{k=1}^{m} \lambda_{kp} \frac{\partial}{\partial \kappa_{km}} + \sum_{k=m+1}^{h} \lambda_{kp} \frac{\partial}{\partial \kappa_{mk}} \right),$$

form a basis for the Lie algebra of left invariant vector fields on $H_R^{(g,h)}$.

THEOREM 1.

$$D_{kl}^0 \Phi_J^{(\mathscr{M})} \begin{bmatrix} A \\ 0 \end{bmatrix} (\Omega \,|\, [(\lambda, \mu), \kappa]) = \pi i \mathscr{M}_{kl} \Phi_J^{(\mathscr{M})} \begin{bmatrix} A \\ 0 \end{bmatrix} (\Omega \,|\, [(\lambda, \mu), \kappa]), \tag{1.8}$$

$$\hat{D}_{mp} \Phi_J^{(\mathscr{M})} \begin{bmatrix} A \\ 0 \end{bmatrix} (\Omega \,|\, [(\lambda, \mu), \kappa]) = 2\pi i \sum_{l=1}^{h} \mathscr{M}_{ml} \Phi_{J+\varepsilon_{lp}}^{(\mathscr{M})} \begin{bmatrix} A \\ 0 \end{bmatrix} (\Omega \,|\, [(\lambda, \mu), \kappa]), \tag{1.9}$$

$$\begin{aligned} D_{mp} \Phi_J^{(\mathscr{M})} \begin{bmatrix} A \\ 0 \end{bmatrix} (\Omega \,|\, [(\lambda, \mu), \kappa]) &= 2\pi i \sum_{l=1}^{h} \sum_{q=1}^{g} \mathscr{M}_{lm} \Omega_{pq} \Phi_{J+\varepsilon_{lq}}^{(\mathscr{M})} \begin{bmatrix} A \\ 0 \end{bmatrix} (\Omega \,|\, [(\lambda, \mu), \kappa]) \\ &\quad + J_{mp} \Phi^{(\mathscr{M})} \begin{bmatrix} A \\ 0 \end{bmatrix} (\Omega \,|\, [(\lambda, \mu), \kappa]). \end{aligned} \tag{1.10}$$

$$(1 \le k \le l \le h, \ 1 \le m \le h, \ 1 \le p \le g)$$

Proof. (1.8) follows immediately from the definition of $\Phi_J^{(\mathscr{M})} \begin{bmatrix} A \\ 0 \end{bmatrix} (\Omega \,|\, [(\lambda, \mu), \kappa])$.

$$\hat{D}_{mp}\Phi_J^{(\mathcal{M})}\begin{bmatrix}A\\0\end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa])$$

$$= -\pi i\sum_{l=1}^{h}\mathcal{M}_{ml}\lambda_{lp}\Phi_J^{(\mathcal{M})}\begin{bmatrix}A\\0\end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa])$$

$$+ 2\pi i\{\pi i\sigma(\mathcal{M}(\kappa - \lambda\,{}^t\mu))\}\sum_{N\in Z^{(h,g)}}(\lambda+N+A)^J\sum_{l=1}^{h}\mathcal{M}_{ml}(\lambda+N+A)_{lp}$$

$$\times \exp\{\pi i\sigma(\mathcal{M}((\lambda+N+A)\Omega\,{}^t(\lambda+N+A) + 2(\lambda+N+A)\,{}^t\mu))\}$$

$$+ \pi i\sum_{l=1}^{h}\mathcal{M}_{ml}\lambda_{lp}\Phi_J^{(\mathcal{M})}\begin{bmatrix}A\\0\end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa])$$

$$= 2\pi i\sum_{l=1}^{h}\mathcal{M}_{ml}\Phi_{J+\varepsilon_{lp}}^{(\mathcal{M})}\begin{bmatrix}A\\0\end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa])\,.$$

We compute

$$\frac{\partial}{\partial\lambda_{mp}}\Phi_J^{(\mathcal{M})}\begin{bmatrix}A\\0\end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa])$$

$$= -\pi i\sum_{k=1}^{h}\mathcal{M}_{km}\mu_{kp}\Phi_J^{(\mathcal{M})}\begin{bmatrix}A\\0\end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa])$$

$$+ 2\pi i\sum_{k=1}^{h}\sum_{q=1}^{g}\mathcal{M}_{km}\Omega_{pq}\Phi_{J+\varepsilon_{kq}}^{(\mathcal{M})}\begin{bmatrix}A\\0\end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa])$$

$$+ J_{mp}\Phi_{J-\varepsilon_{mp}}^{(\mathcal{M})}\begin{bmatrix}A\\0\end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa])$$

$$+ 2\pi i\sum_{k=1}^{h}\mathcal{M}_{km}\mu_{kp}\Phi_J^{(\mathcal{M})}\begin{bmatrix}A\\0\end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa])\,.$$

Therefore we obtain (1.8) and (1.10). q.e.d.

COROLLARY 1.4.

$$\left(D_{mp} - \sum_{q=1}^{g}\Omega_{pq}\hat{D}_{mq}\right)\Phi_J^{(\mathcal{M})}\begin{bmatrix}A\\0\end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa]) = J_{mp}\Phi_{J-\varepsilon_{mp}}^{(\mathcal{M})}\begin{bmatrix}A\\0\end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa])\,.$$

Let $H_\Omega^{(\mathcal{M})}\begin{bmatrix}A\\0\end{bmatrix}$ *be the completion of the vector space spanned by* $\Phi_J^{(\mathcal{M})}\begin{bmatrix}A\\0\end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa])$ $(J\in Z_{\geq}^{(h,g)})$ *and let* $\overline{H_\Omega^{(\mathcal{M})}\begin{bmatrix}A\\0\end{bmatrix}}$ *be the complex conjugate of* $H_\Omega^{(\mathcal{M})}\begin{bmatrix}A\\0\end{bmatrix}$.

THEOREM 2. $H_\Omega^{(\mathcal{M})}\begin{bmatrix}A\\0\end{bmatrix}$ *and* $\overline{H_\Omega^{(\mathcal{M})}\begin{bmatrix}A\\0\end{bmatrix}}$ *are irreducible invariant subspaces of* $L^2(H_Z^{(h,g)}\backslash H_R^{(h,g)})$ *with respect to the right regular representation* ρ. *In addition, we have*

$$H_{\Omega}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix} = \exp\{2\pi i\sigma(\mathcal{M}\mu\, {}^t A)\}H_{\Omega}^{(\mathcal{M})}\begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\rho([(0,0),\kappa])\phi = \exp\{\pi i\sigma(\mathcal{M}\kappa)\}\phi \qquad \left(\phi \in H_{\Omega}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}\right),$$

$$\rho([(0,0),\kappa])\bar{\phi} = \exp\{-\pi i\sigma(\mathcal{M}\kappa)\}\bar{\phi} \qquad \left(\bar{\phi} \in \overline{H_{\Omega}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}}\right).$$

Proof. It follows from Theorem 1, Proposition 1.3 and the definition of $\Phi_{\mathcal{J}}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa])$. q.e.d.

§ 2. Proof of the Main Theorem

We fix an element $\Omega \in H_g$ once and for all. We introduce a system of complex coordinates with respect to Ω:

$$Z = \mu + \lambda\Omega\,, \quad \bar{Z} = \mu + \lambda\bar{\Omega}\,, \quad \lambda, \mu \text{ real}. \tag{2.1}$$

We set

$$dZ = \begin{pmatrix} dZ_{11} & \cdots & dZ_{1g} \\ \vdots & \ddots & \vdots \\ dZ_{h1} & \cdots & dZ_{hg} \end{pmatrix}, \qquad \frac{\partial}{\partial Z} = \begin{pmatrix} \frac{\partial}{\partial Z_{11}} & \cdots & \frac{\partial}{\partial Z_{h1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial Z_{1g}} & \cdots & \frac{\partial}{\partial Z_{hg}} \end{pmatrix}.$$

Then an easy computation yields

$$\frac{\partial}{\partial \lambda} = \Omega\frac{\partial}{\partial Z} + \bar{\Omega}\frac{\partial}{\partial \bar{Z}}\,,$$

$$\frac{\partial}{\partial \mu} = \frac{\partial}{\partial Z} + \frac{\partial}{\partial \bar{Z}}\,.$$

Thus we obtain the following

$$\frac{\partial}{\partial \bar{Z}} = \frac{i}{2}(\operatorname{Im}\Omega)^{-1}\left(\frac{\partial}{\partial \lambda} - \Omega\frac{\partial}{\partial \mu}\right). \tag{2.2}$$

LEMMA 2.1.

$$\Phi_{\Omega}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa])$$
$$= \exp\{\pi i\sigma(\mathcal{M}(\lambda\Omega\,{}^t\lambda + \lambda\,{}^t\mu + \kappa))\}\vartheta_{\mathcal{J}}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,\lambda, \mu + \lambda\Omega)\,.$$

Proof. It follows immediately from (1.4) and (1.5).

LEMMA 2.2. *Let $\Phi([(\lambda, \mu), \kappa])$ be a real analytic function on $H_Z^{(g,h)} \backslash H_R^{(g,h)}$ such that*

i) $\exp\{-\pi i\sigma(\mathcal{M}\kappa)\}\Phi([(\lambda, \mu), \kappa])$ *is independent of* κ,

ii) $(D_{mp} - \sum_{q=1}^{g} \Omega_{pq}\hat{D}_{mq})\Phi = 0$ *for all* $1 \leq m \leq h$ *and* $1 \leq p \leq g$, *where* $\mathcal{M}$ *is a positive definite symmetric even integral matrix of degree h. Let*

$$\Psi(\lambda, \mu) = \exp\{-\pi i\sigma(\mathcal{M}(\lambda\Omega\,{}^t\lambda + \lambda\,{}^t\lambda + \kappa))\}\Phi([(\lambda, \mu), \kappa]) . \tag{2.3}$$

Then $\Psi(\lambda, \mu)$ is a mixed theta function of level $\mathcal{M}$ in $Z = \mu + \lambda\Omega$ with respect to Ω.

Proof. By the assumption (i), we have

$$\begin{aligned}&\Psi(\lambda + \tilde{\lambda}, \mu + \tilde{\mu})\\ &\quad= \exp\{-\pi i\sigma(\mathcal{M}((\lambda + \tilde{\lambda})\Omega\,{}^t(\lambda + \tilde{\lambda}) + (\lambda + \tilde{\lambda})\,{}^t(\mu + \tilde{\mu}) + \kappa + \tilde{\kappa} + \tilde{\lambda}\,{}^t\mu - \tilde{\mu}\,{}^t\lambda))\}\\ &\qquad\qquad \Phi([(\tilde{\lambda}, \tilde{\mu}), \tilde{\kappa}] \circ [(\lambda, \mu), \kappa])\\ &\qquad\qquad\quad = \exp\{-\pi i\sigma(\mathcal{M}(\tilde{\lambda}\Omega\,{}^t\tilde{\lambda} + 2(\mu + \lambda\Omega)\,{}^t\tilde{\lambda}))\}\Psi(\lambda, \mu) ,\end{aligned}$$

where $[(\tilde{\lambda}, \tilde{\mu}), \tilde{\kappa}] \in H_Z^{(g,h)}$. In the last equality, we used the facts that $\sigma(\mathcal{M}(\tilde{\kappa} + \tilde{\lambda}\,{}^t\tilde{\mu})) \in 2Z$ because $\tilde{\kappa} + \tilde{\mu}\,{}^t\tilde{\lambda}$ is symmetric. This implies that $\Psi(\lambda, \mu)$ satisfies the condition (2) in Definition 1.2. Now we must show that $\Psi(\lambda, \mu)$ is holomorphic in $Z = \mu + \lambda\Omega$, that is,

$$\frac{\partial\Psi}{\partial\bar{Z}} = 0 , \qquad Z = \mu + \lambda\Omega . \tag{2.4}$$

By (2.2) the equation (2.4) is equivalent to the equation

$$\left(\frac{\partial}{\partial\lambda_{mp}} - \sum_{q=1}^{g} \Omega_{pq}\frac{\partial}{\partial\mu_{mq}}\right)\Psi(\lambda, \mu) = 0 , \quad 1 \leq m \leq h , \quad 1 \leq p \leq g . \tag{2.5}$$

But according to (1.9) and (1.10), we have

$$\frac{\partial}{\partial\lambda_{mp}} - \sum_{q=1}^{g} \Omega_{pq}\frac{\partial}{\partial\mu_{mq}} = D_{mp} - \sum_{q=1}^{g} \Omega_{pq}\hat{D}_{mq} + P ,$$

where

$$\begin{aligned}P = \sum_{k=1}^{m} \mu_{kp}D_{km}^0 + \sum_{k=m+1}^{h} \mu_{kp}D_{mk}^0 - \sum_{k=1}^{m}\sum_{q=1}^{g} \Omega_{pq}\lambda_{kq}D_{km}^0\\ - \sum_{k=m+1}^{h}\sum_{q=1}^{g} \Omega_{pq}\lambda_{kq}D_{mk}^0 .\end{aligned}$$

We observe that $P\cdot\Psi(\lambda, \mu) = 0$ because $\Psi(\lambda, \mu)$ is independent of κ by the assumption (i). We let

$$f([(\lambda, \mu), \kappa]) = \exp\{-\pi i\sigma(\mathcal{M}(\lambda\Omega\,{}^t\lambda + \lambda\,{}^t\mu + \kappa))\}.$$

Then $\Psi(\lambda, \mu) = f([(\lambda, \mu), \kappa])\Phi([(\lambda, \mu), \kappa])$. Then in order to show that $\Psi(\lambda, \mu)$ is holomorphic in the complex variables $Z = \mu + \lambda\Omega$ with respect to Ω, by the assumption (ii), it suffices to show the following:

$$\left(D_{mp} - \sum_{q=1}^{g} \Omega_{pq}\hat{D}_{mq}\right) f([(\lambda, \mu), \kappa]) = 0. \tag{2.6}$$

By an easy computation, we obtain (2.6). This completes the proof of Lemma 2.2. q.e.d.

The Stone-von Neumann theorem says that an irreducible representation ρ_c of $H_R^{(g,h)}$ is characterized by a real symmetric matrix $c \in R^{(h,h)}$ $(c \neq 0)$ such that

$$\rho_c([(\lambda, \mu), \kappa]) = \exp\{\pi i\sigma(c\kappa)\}I, \qquad \kappa = {}^t\kappa \in R^{(h,h)}, \tag{2.7}$$

where I denotes the identity map of the representation space. If $c = 0$, it is characterized by a pair $(k, m) \in R^{(h,g)} \times R^{(h,g)}$ such that

$$\rho_{k,m}([(\lambda, \mu), \kappa]) = \exp\{2\pi i\sigma(k\,{}^t\lambda + m\,{}^t\mu)\}I. \tag{2.8}$$

If $\Phi \in L^2(H_Z^{(g,h)}\backslash H_R^{(g,h)})$ and $\tilde{\kappa} = {}^t\tilde{\kappa} \in Z^{(h,h)}$, then

$$\begin{aligned}
\Phi([(\lambda, \mu), \kappa]) &= \Phi([(0, 0), \tilde{\kappa}] \circ [(\lambda, \mu), \kappa]) \\
&= \Phi([(\lambda, \mu), \kappa] \circ [(0, 0), \tilde{\kappa}]) \\
&= \rho_c([(0, 0), \tilde{\kappa}])\Phi([(\lambda, \mu), \kappa]) \\
&= \exp\{\pi i\sigma(c\tilde{\kappa})\}\Phi([(\lambda, \mu), \kappa]).
\end{aligned}$$

Thus if $c \neq 0$, $\sigma(c\tilde{\kappa}) \in 2Z$ for all $\tilde{\kappa} = {}^t\tilde{\kappa} \in Z^{(h,h)}$. It means that ${}^tc = c = (c_{ij})$ must be even integral, that is, all diagonal elements c_{ii} $(1 \leq i \leq h)$ are even integers and all c_{ij} $(i \neq j)$ are integers. If $c = 0$, $\sigma(k\,{}^t\lambda + m\,{}^t\mu) \in Z$ for all $\lambda, \mu \in Z^{(h,g)}$ and hence $k, m \in Z^{(h,g)}$. Therefore only the irreducible representation $\rho_{\mathcal{M}}$ with $\mathcal{M} = {}^t\mathcal{M}$ even integral and $\rho_{k,m}$ $(k, m \in Z^{(h,g)})$ could occur in the right regular representation ρ in $L^2(H_Z^{(g,h)}\backslash H_R^{(g,h)})$.

Now we prove

Main Theorem. *Let $\mathcal{N} \neq 0$ be an even integral matrix of degree h which is neither positive nor negative definite. Let $R(\mathcal{N})$ be the sum of irreducible representations $\rho_{\mathcal{N}}$ which occur in the right regular representation ρ of $H_R^{(g,h)}$. Let $H_\Omega^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}$ be defined in Theorem 2 for a positive definite even integral matrix $\mathcal{M} > 0$. Then the decomposition of the right regular representation ρ is given by*

$$L^2(H_Z^{(g,h)}\backslash H_R^{(g,h)}) = \bigoplus_{\mathcal{M},A} H_\Omega^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix} \oplus \overline{\left(\bigoplus_{\mathcal{M},A} H_\Omega^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}\right)} \oplus \left(\bigoplus_{\mathcal{N}} R(\mathcal{N})\right)$$
$$\oplus \Big(\bigoplus_{(k,m)\in Z^{(h,g)}} C \exp\{2\pi i\sigma(k\,{}^t\lambda + m\,{}^t\mu)\}\Big) .$$

where $\mathcal{M}$ (resp. $\mathcal{N}$) *runs over the set of all positive definite symmetric, even integral matrices of degree h* (resp. *the set of all even integral nonzero matrices of degree h which are neither positive nor negative definite*) *and A runs over a complete system of representatives of the cosets* $\mathcal{M}^{-1}Z^{(h,g)}/Z^{(h,g)}$. $H_\Omega^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}$ *and* $\overline{H_\Omega^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}}$ *are irreducible invariant subspaces of* $L^2(H_Z^{(g,h)}\backslash H_R^{(g,h)})$ *such that*

$$\rho([(0,0),\tilde{\kappa}])\phi([(\lambda,\mu),\kappa]) = \exp\{\pi i\sigma(\mathcal{M}\tilde{\kappa})\}\phi([(\lambda,\mu),\kappa]) ,$$
$$\rho([(0,0),\tilde{\kappa}])\overline{\phi([(\lambda,\mu),\kappa])} = \exp\{-\pi i\sigma(\mathcal{M}\tilde{\kappa})\}\overline{\phi([(\lambda,\mu),\kappa])}$$

for all $\phi\in H_\Omega^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}$. *And we have*

$$H_\Omega^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix} = \exp\{2\pi i\sigma(\mathcal{M}\mu\,{}^tA)\} H_\Omega^{(\mathcal{M})}\begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This result generalizes that of H. Morikawa ([M]).

Proof. Let $\mathcal{A}$ be the space of real analytic functions on $L^2(H_Z^{(g,h)}\backslash H_R^{(g,h)})$. Since $\mathcal{A}$ is dense in $L^2(H_Z^{(g,h)}\backslash H_R^{(g,h)})$ and $\mathcal{A}$ is invariant under ρ, it suffices to decompose $\mathcal{A}$. Let W be an irreducible invariant subspace of $\mathcal{A}$ such that $\rho([(0,0),\tilde{\kappa}])w = \exp\{2\pi i\sigma(\mathcal{M}\tilde{\kappa})\}w$ for all $w\in W$, where $\mathcal{M} = {}^t\mathcal{M}$ is a positive definite even integral matrix of degree h. Then W is isomorphic to $H_\Omega^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}\cap\mathcal{A}$ for some $A\in\mathcal{M}^{-1}Z^{(h,g)}/Z^{(h,g)}$ and $\Omega\in H_g$. Since $H_\Omega^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}\cap\mathcal{A}$ contains an element $\Phi_0^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa])$ (see Corollary 1.4) satisfying

$$\Big(D_{mp} - \sum_{q=1}^{g}\Omega_{pq}\hat{D}_{mq}\Big)\Phi_0^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa]) = 0$$

for all $1\le m\le h$, $1\le p\le g$, there exists an element $\Phi_0([(\lambda,\mu),\kappa])$ in W such that

$$\Big(D_{mp} - \sum_{q=1}^{g}\Omega_{pq}\hat{D}_{mq}\Big)\Phi_0([(\lambda,\mu),\kappa]) = 0$$

for all $1\le m\le h$, $1\le p\le g$. On the other hand, we have

$$\begin{aligned}\Phi_0([(\lambda, \mu), \kappa]) &= \rho([(0, 0), \kappa])\Phi_0([(\lambda, \mu), 0]) \\ &= \exp\{\pi i\sigma(\mathcal{M}\kappa)\}\Phi_0([(\lambda, \mu), 0]) .\end{aligned}$$

Therefore $\Phi_0([(\lambda, \mu), \kappa])$ satisfies the conditions of Lemma 2. Thus we have

$$\begin{aligned}\Phi_0([(\lambda, \mu), \kappa]) &= \exp\{\pi i\sigma(\mathcal{M}(\lambda\Omega\, {}^t\lambda + \lambda\, {}^t\mu + \kappa))\} \sum_{A, J} \alpha_{AJ}\vartheta_J^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega \mid \lambda, \mu + \lambda\Omega) \\ &= \sum_{A, J} \alpha_{AJ}\Phi_J^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega \mid [(\lambda, \mu), \kappa]) \qquad \text{(by Lemma 2.1)} ,\end{aligned}$$

where A (resp. J) runs over $\mathcal{M}^{-1}Z^{(h,g)}/Z^{(h,g)}$ (resp. $Z_{\geq 0}^{(h,g)}$). Hence $\Phi_0 \in \bigoplus_A H_\Omega^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}$. By the way, since W is spanned by $D_{kl}^0\Phi_0$, $D_{mp}\Phi_0$ and $\hat{D}_{mp}\Phi_0$, we have $W \subset \bigoplus_A H_\Omega^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}$. So $W = H_\Omega^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix} \cap \mathcal{A}$ for some $A \in \mathcal{M}^{-1}Z^{(h,g)}/Z^{(h,g)}$. Similarly, $\overline{W} = \overline{H_\Omega^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}} \cap \mathcal{A}$. Clearly for each $(k, m) \in Z^{(h,g)} \times Z^{(h,g)}$,

$$W_{k,m} := C \exp\{2\pi i(k\, {}^t\lambda + m\, {}^t\mu)\}$$

is a one dimensional irreducible invariant subspace of $L^2(H_Z^{(h,g)} \backslash H_R^{(g,h)})$. The latter part of the above theorem is the restatement of Theorem 2. This completes the main theorem. q.e.d.

COROLLARY. *For even integral matrix* $\mathcal{M} = {}^t\mathcal{M} > 0$ *of degree* h, *the multiplicity* $m_{\mathcal{M}}$ *of* $\rho_{\mathcal{M}}$ *in* ρ *is given by*

$$m_{\mathcal{M}} = (\det \mathcal{M})^g .$$

CONJECTURE. *For any even integral matrix* $\mathcal{N} \neq 0$ *of degree* h *which is neither positive nor negative definite, the multiplicity* $m_{\mathcal{N}}$ *of* $\rho_{\mathcal{N}}$ *in* ρ *is a zero, that is,* $R(\mathcal{N})$ *vanishes.*

§ 3. Schrödinger representations

Let $\Omega \in H_g$ and let $\mathcal{M} = {}^t\mathcal{M}$ be a positive definite even integral matrix of degree h. We set $\Omega = \Omega_1 + i\Omega_2$ ($\Omega_1, \Omega_2 \in R^{(g,g)}$). Let $L^2(R^{(h,g)}, \mu_{\Omega_2}^{(\mathcal{M})})$ be the L^2-space of $R^{(h,g)}$ with respect to the measure

$$\mu_{\Omega_2}^{(\mathcal{M})}(d\xi) = \exp\{-2\pi\sigma(\mathcal{M}\xi\Omega_2\, {}^t\xi)\}d\xi .$$

It is easy to show that the transformation $f(\xi) \mapsto \exp\{\pi i\sigma(\mathcal{M}\xi\Omega_2\, {}^t\xi)\}f(\xi)$ of $L^2(R^{(h,g)}, \mu_{\Omega_2}^{(\mathcal{M})})$ into $L^2(R^{(h,g)}, d\xi)$ is an isomorphism. Since the set $\{\xi^J \mid J \in Z_{\geq 0}^{(h,g)}\}$ is a basis of $L^2(R^{(h,g)}, \mu_{\Omega_2}^{(\mathcal{M})})$, the set $\{\exp(\pi i\sigma(\mathcal{M}\xi\Omega\,{}^t\xi))\xi^J \mid J \in Z_{\geq 0}^{(h,g)}\}$ is a basis of $L^2(R^{(h,g)}, d\xi)$.

LEMMA 3.1.

$$\left\langle \Phi_J^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa]),\ \Phi_K^{(\tilde{\mathcal{M}})}\begin{bmatrix} \tilde{A} \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa])\right\rangle$$

$$= \int_{H_Z^{(g,h)}\backslash H_R^{(g,h)}} \Phi_J^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa])\cdot\overline{\Phi_K^{(\tilde{\mathcal{M}})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[\lambda,\mu),\kappa])}\,d\lambda d\mu d\kappa$$

$$= \begin{cases} \displaystyle\int_{R^{(h,g)}} y^{J+K}\exp\{-2\pi\sigma(\mathcal{M}y\Omega_2\,{}^t y)\}dy & \text{if } \mathcal{M}=\tilde{\mathcal{M}},\ A\equiv\tilde{A}\ (\mathrm{mod}\,\mathcal{M}), \\ 0, & \text{otherwise}. \end{cases}$$

It is easy to prove the above lemma and so we omit its proof. According to the above argument and Lemma 3.1, we obtain the following:

LEMMA 3.2. *The transformation of* $L^2(R^{(h,g)}, \mu_{\Omega_2}^{(\mathcal{M})})$ *onto* $H_\Omega^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}$ *given by*

$$\xi^J \longmapsto \Phi_J^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa]),\qquad J\in Z_{\geq 0}^{(h,g)} \tag{3.1}$$

is an isomorphism of Hilbert spaces.

Now we define a unitary representation of $H_R^{(h,g)}$ on $L^2(R^{(h,g)}, d\xi)$ by

$$U_{\mathcal{M}}([(\lambda,\mu),\kappa])f(\xi) = \exp\{-\pi i\sigma(\mathcal{M}(\kappa+\mu\,{}^t\lambda+2\mu\,{}^t\xi))\}f(\xi+\lambda), \tag{3.2}$$

where $[(\lambda,\mu),\kappa]\in H_R^{(g,h)}$ and $f\in L^2(R^{(h,g)}, d\xi)$. $U_{\mathcal{M}}$ is called the *Schrödinger representation* of $H_R^{(h,g)}$ of *index* $\mathcal{M}$.

PROPOSITION 3.3. *If we set* $f_J(\xi) = \exp\{\pi i\sigma(\mathcal{M}\xi\Omega\,{}^t\xi)\}\xi^J$ $(J\in Z_{\geq 0}^{(h,g)})$, *we have*

$$dU_{\mathcal{M}}(D_{kl}^0)f_J(\xi) = -\pi i\mathcal{M}_{kl}f_J(\xi),\qquad 1\leq k\leq l\leq h. \tag{3.3}$$

$$dU_{\mathcal{M}}(D_{mp})f_J(\xi) = 2\pi i\sum_{l=1}^{h}\sum_{q=1}^{g}\mathcal{M}_{ml}\Omega_{pq}f_{J+\varepsilon_{lq}}(\xi) + J_{mp}f_{J-\varepsilon_{mp}}(\xi). \tag{3.4}$$

$$dU_{\mathcal{M}}(\hat{D}_{mp})f_J(\xi) = -\pi i\sum_{l=1}^{h}\mathcal{M}_{ml}f_{J+\varepsilon_{lp}}(\xi). \tag{3.4}$$

Proof.

$$\begin{aligned} dU_{\mathcal{M}}(D_{kl}^0)f_J(\xi) &= \frac{d}{dt}\Big|_{t=0} U_{\mathcal{M}}(\exp(tX_{kl}^0))f_J(\xi) \\ &= \frac{d}{dt}\Big|_{t=0} U_{\mathcal{M}}([(0,0),tE_{kl}^0])_J(\xi) \\ &= \lim_{t\to 0}\frac{\exp\{-\pi i\sigma(t\mathcal{M}E_{kl}^0)\}-I}{t}f_J(\xi) \\ &= -\pi i\mathcal{M}_{kl}f_J(\xi). \end{aligned}$$

$$
\begin{aligned}
dU_{\mathcal{M}}(D_{mp})f_J(\xi) &= \frac{d}{dt}\Big|_{t=0} U_{\mathcal{M}}(\exp(tX_{mp}))f_J(\xi) \\
&= \frac{d}{dt}\Big|_{t=0} U_{\mathcal{M}}([(tE_{mp}, 0), 0])f_J(\xi) \\
&= \frac{d}{dt}\Big|_{t=0} \exp\{\pi i\sigma(\mathcal{M}(\xi + {}^tE_{mp})\Omega\, {}^t(\xi + tE_{mp}))\}(\xi + tE_{mp})^J \\
&= 2\pi i \sum_{l=1}^{h}\sum_{q=1}^{g} \mathcal{M}_{ml}\Omega_{pq} f_{J+\varepsilon_{lq}}(\xi) + J_{mp} f_{J-\varepsilon_{mp}}(\xi)\,.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
dU_{\mathcal{M}}(\hat{D}_{mp})f_J(\xi) &= \frac{d}{dt}\Big|_{t=0} U_{\mathcal{M}}(\exp(t\hat{X}_{mp}))f_J(\xi) \\
&= \frac{d}{dt}\Big|_{t=0} U_{\mathcal{M}}([(0, tE_{mp}), 0])f_J(\xi) \\
&= \lim_{t\to 0} \frac{\exp\{-2\pi i\sigma(t\mathcal{M}E_{mp}{}^t\xi)\} - I}{t} f_J(\xi) \\
&= -\pi i \sum_{l=1}^{h} \mathcal{M}_{ml} f_{J+\varepsilon_{lp}}(\xi)\,.
\end{aligned}
$$

q.e.d.

THEOREM 3. *Let* $\Phi_{\Omega}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}$ *be the transform of* $L^2(R^{(h,g)}, d\xi)$ *onto* $H_{\Omega}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}$ *defined by*

$$
\begin{aligned}
\text{(3.6)}\qquad & \Phi_{\Omega}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\exp(\pi i\sigma(\mathcal{M}\xi\Omega\,{}^t\xi))\xi^J)([(\lambda, \mu), \kappa]) \\
&= \Phi_J^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega \,|\, [(\lambda, \mu), \kappa])\,, \qquad J \in Z_{\geq 0}^{(h,g)}\,.
\end{aligned}
$$

Then $\Phi_{\Omega}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}$ *is an isomorphism of the Hilbert space* $L^2(R^{(h,g)}, d\xi)$ *onto the Hilbert space* $H_{\Omega}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}$ *such that*

$$
\text{(3.7)}\qquad \hat{\rho}([(\lambda, \mu), \kappa]) \circ \Phi_{\Omega}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix} = \Phi_{\Omega}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix} \circ U_{\mathcal{M}}([(\lambda, \mu), \kappa])\,,
$$

$$
\text{(3.8)}\qquad \Phi_{\Omega}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix} = \exp\{2\pi i\sigma(\mathcal{M}A\,{}^t\mu)\}\rho([(A, 0), 0])\Phi_{\Omega}^{(\mathcal{M})}\begin{bmatrix} 0 \\ 0 \end{bmatrix},
$$

where $\hat{\rho}$ *is the unitary representation of* $H_R^{(g,h)}$ *on* $H_{\Omega}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}$ *defined by*

$$
\hat{\rho}([(\lambda, \mu), \kappa])\phi = \rho([(\lambda, -\mu), -\kappa])\phi\,, \qquad \phi \in H_{\Omega}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}.
$$

Proof. For brevity, we set $f_J(\xi) = \exp\{\pi i\sigma(\mathcal{M}\xi\Omega\,{}^t\xi)\}\xi^J$ $(J \in Z_{\geq 0}^{(h,g)})$. Using Proposition 3.3, we obtain

$$\begin{aligned}
&\Phi_\Omega^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(dU_{\mathcal{M}}(-D_{kl}^0)(f_J(\xi)))([(\lambda, \mu), \kappa]) \\
&= \pi i\mathcal{M}_{kl}\Phi_\Omega^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(f_J(\xi))([(\lambda, \mu), \kappa]) \\
&= \pi i\mathcal{M}_{kl}\Phi_J^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda, \mu), \kappa]) \\
&= d\rho(D_{kl}^0)\left\{\Phi_\Omega^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(f_J(\xi))[(\lambda, \mu), \kappa]\right\}.
\end{aligned}$$

$$\begin{aligned}
&\Phi_\Omega^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(dU_{\mathcal{M}}(D_{mp})(f_J(\xi)))([(\lambda, \mu), \kappa]) \\
&= 2\pi i\sum_{l=1}^{h}\sum_{q=1}^{g}\mathcal{M}_{ml}\Omega_{pq}\Phi_\Omega^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(f_{J+\varepsilon_{lq}}(\xi))([(\lambda, \mu), \kappa]) \\
&\quad + J_{mp}\Phi_\Omega^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(f_{J-\varepsilon_{mp}}(\xi))([(\lambda, \mu), \kappa]) \\
&= 2\pi i\sum_{l=1}^{h}\sum_{q=1}^{g}\mathcal{M}_{ml}\Omega_{pq}\Phi_{J+\varepsilon_{lq}}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda, \mu), \kappa]) \\
&\quad + J_{mp}\Phi_{J-\varepsilon_{mp}}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda, \mu), \kappa]) \\
&= d\rho(D_{mp})\Phi_J^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda, \mu), \kappa]) \\
&= d\rho(D_{mp})\left\{\Phi_\Omega^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(f_J(\xi))([(\lambda, \mu), \kappa])\right\}.
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
&\Phi_\Omega^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(dU_{\mathcal{M}}(-\hat{D}_{mp}(f_J(\xi)))([(\lambda, \mu), \kappa]) \\
&= \pi i\sum_{l=1}^{h}\mathcal{M}_{ml}\Phi_\Omega^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(f_{J+\varepsilon_{lp}}(\xi))([(\lambda, \mu), \kappa]) \\
&= \pi i\sum_{l=1}^{h}\mathcal{M}_{ml}\Phi_{J+\varepsilon_{lp}}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda, \mu), \kappa]) \\
&= d\rho(\hat{D}_{mp})\Phi_J^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda, \mu), \kappa]) \\
&= d\rho(\hat{D}_{mp})\left\{\Phi_\Omega^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(f_J(\xi))([(\lambda, \mu), \kappa])\right\},
\end{aligned}$$

where $1 \leq k \leq l \leq h$, $1 \leq p \leq g$. The last statement is obvious. q.e.d.

Remark 3.4. Theorem 3 means that the unitary representation $\hat{\rho}$ of $H_{\mathbb{R}}^{(g,h)}$ on $H_{\Omega}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}$ is equivalent to the Schrödinger representation $U_{\mathcal{M}}$ of index $\mathcal{M}$. Thus the Schrödinger representation $U_{\mathcal{M}}$ is irreducible.

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